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Nonlinear dynamics

From laboratories to astrophysics:

The expanding universe of plasma physics

Les Houches, France

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Introductory remarks

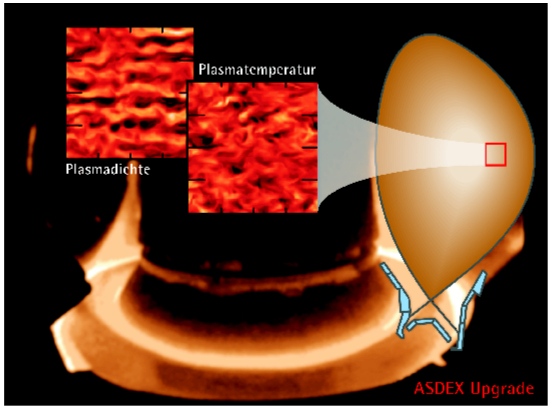
This lecture is meant to be an introduction
to turbulence in fluids and plasmas;
no prior knowledge of turbulence is required

You are invited to ask questions at any time

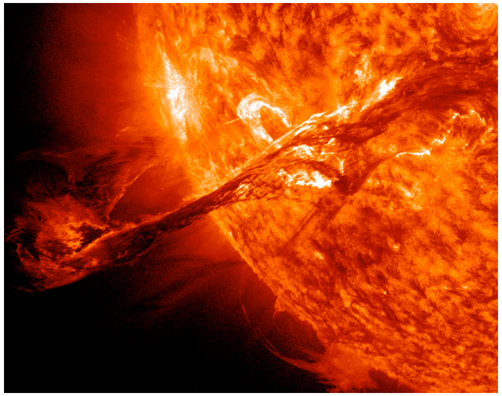
Lectures may explain, motivate, inspire etc.,
but they cannot replace individual study...

Turbulence in lab & natural plasmas

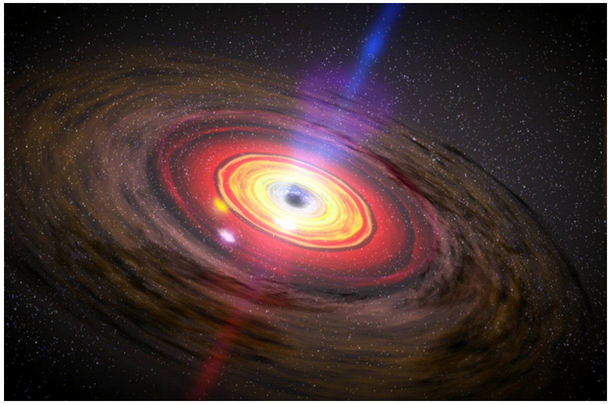
Fusion plasmas



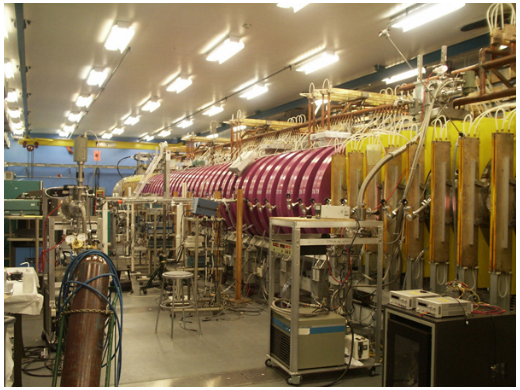
Space plasmas



Astrophysical plasmas



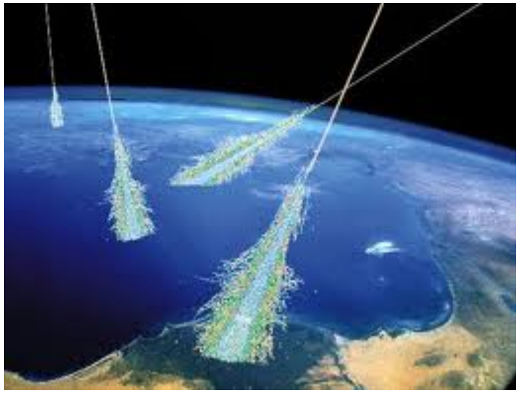
Basic plasma science



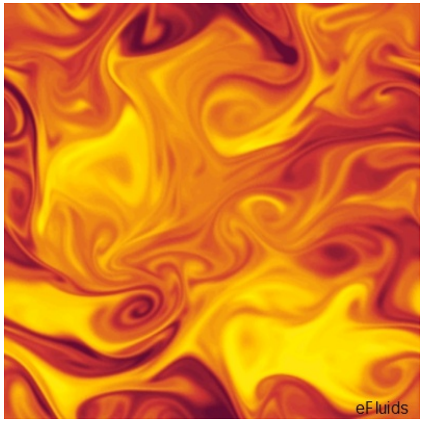
What are the fundamental principles of plasma turbulence?

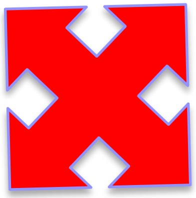
Central role of plasma turbulence

Particle acceleration & propagation

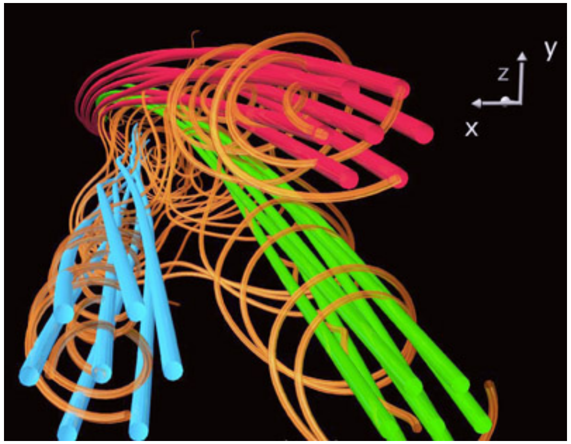


Cross-field transport

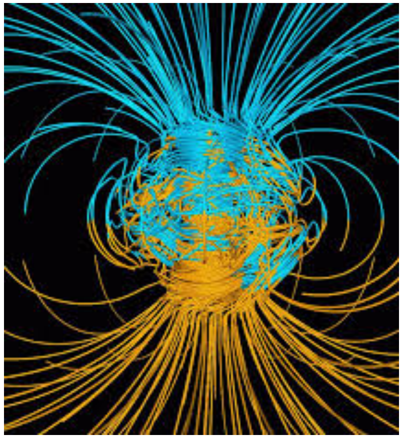




Magnetic reconnection



Dynamo action

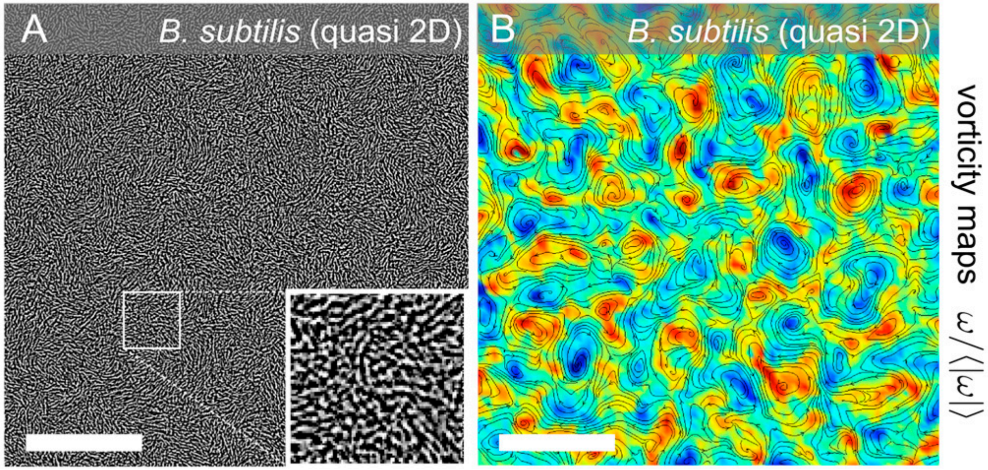


What is the role of plasma turbulence in these processes?

Some turbulence basics

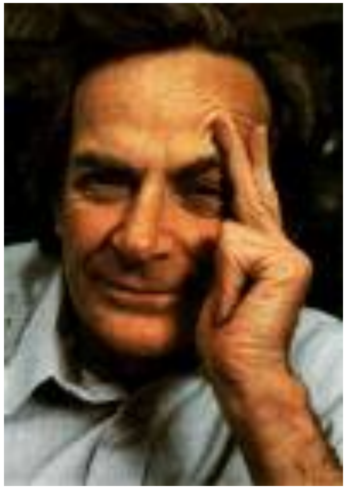
Turbulence is ubiquitous…

Active fluids (dense bacterial suspensions)



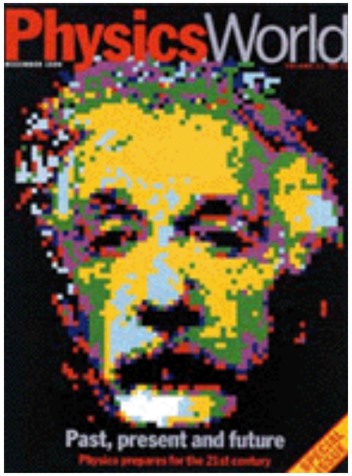
…as well as an important unsolved physics problem

According to a famous statement by Richard Feynman…



…and a survey by the British "Institute of Physics" among many of the leading physicists world-wide…

"Millennium Issue"
(December 1999)

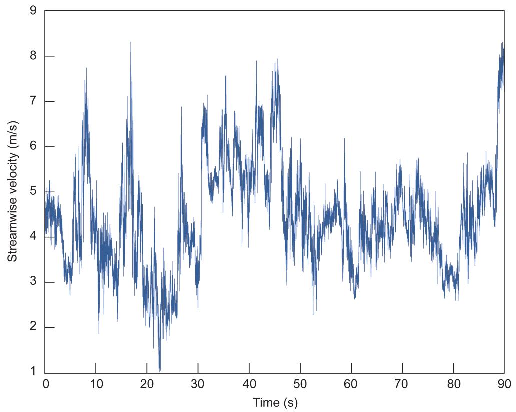


TURBULENCE:
A challenging topic for both basic and applied research

WHAT IS TURBULENCE THEN?

Turbulence…

- is an intrinsically nonlinear phenomenon
- occurs only in open systems
- involves many degrees of freedom
- is highly irregular in space and time
- often leads to a statistically quasi-stationary state far from thermodynamic equilibrium



Note that these are also the features of LIFE!

WHAT IS THE CHALLENGE?

Theories that **don't** apply directly:

- nonlinear dynamics
- equilibrium statistical mechanics
- nonequilibrium statistical mechanics near equilibrium

Co-existence of **randomness** and **coherence**

Two dangers constantly threaten the world:
order and **disorder**. Paul Valéry

VISION BY J. VON NEUMANN

Supercomputers can help to unravel the “mysteries” of turbulence in the spirit of John von Neumann (1949):

„There might be some hope to 'break the deadlock' by extensive, but well-planned, computational efforts. There are strong indications that one could name certain strategic points in this complex, where relevant information must be obtained by direct calculations. This should, in the end, make an attack with analytical methods possible.“

Milestones in turbulence research I

Osborne Reynolds (1842-1912)

1883 Transition from laminar to turbulent flows (Reynolds number)

1895 Reynolds decomposition into mean and fluctuating flows

Ludwig Prandtl (1875-1953)

1904 Recognition of the importance of boundary layers

1925 Mixing length model for turbulent transport

Lewis Fry Richardson (1881-1953)

1922 Book „Weather Prediction by Numerical Process“

Notion of turbulent eddies & (local, direct) energy cascade

Geoffrey Ingram Taylor (1886-1975)

1935 Series of papers on the „Statistical Theory of Turbulence“

Milestones in turbulence research II

Werner Heisenberg (1901-1976)

1923 „Über Stabilität und Turbulenz von Flüssigkeitsströmen"
(PhD Thesis, LMU München, Advisor: Arnold Sommerfeld)

1948 Three papers on the statistical theory of turbulence

Andrey Kolmogorov (1903-1987)

1941 K41 theory: dimensional analysis, -5/3 law (energy spectrum)

1962 K62 theory: scale invariance is broken, problem of intermittency

Robert Kraichnan (1928-2008)

1957- Field-theoretic approach: *Direct Interaction Approximation*

1967 Inverse energy cascade in two-dimensional fluid turbulence

1973 Field-theoretic approach: *Martin-Siggia-Rose formalism*

Milestones in turbulence research III

Steven Orszag (1943-2011)

1966 *Eddy-Damped Quasi-Normal Markovian* approximation

1969- Towards *Direct Numerical Simulations* via spectral methods

1972 First 3D DNS on a 32^3 grid by S. Orszag & G. Patterson

1948- *Numerical weather prediction* by J. von Neumann & J. Charney

1963 *Large-Eddy Simulation* techniques by J. Smagorinsky

1965 *Fast Fourier Transform* algorithm by J. Cooley & J. Tukey

1977 Cray-1 at the *National Center for Atmospheric Research*

2002 First 3D DNS on a 4096^3 grid by Y. Kaneda et al.

Turbulence: The bigger picture

Some grand challenges

- Design airplanes, ships, cars etc.
- Predict weather & climate
- Unravel role of turbulence in space & astrophysics
- Predict performance of fusion devices like ITER

Some open problems beyond *Homogeneous Isotropic Turbulence*

- Effects of inhomogeneity, anisotropy, compressibility
  (role of walls, drive, stratification, rotation etc.)
- From fluid to magneto-/multi-fluid to kinetic turbulence

Conceptual approach

- Ab initio simulations of complicated problems are not feasible
- Seek physics understanding to construct reliable minimal models

On the physics of 3D fluid turbulence

Early observations

Leonardo da Vinci (ca. 1500)

The Navier-Stokes equations (1822)

The NSE for incompressible fluids:

v: kinematic viscosity

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nu \nabla^2 \boldsymbol{v},$$

$$\nabla \cdot \boldsymbol{v} = 0.$$

p: pressure / mass density

Expressed in terms of vorticity $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{v}$:

$$\partial_t \boldsymbol{\omega} = \nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}$$

$$\nabla v^2 = 2\boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\boldsymbol{v} \wedge (\nabla \wedge \boldsymbol{v})$$

Claude Navier (1785-1836)

George Stokes (1819-1903)

SELF-SIMILARITY & PIPE FLOWS

Osborne Reynolds (1883)

$$\mathbf{v} = V\tilde{\mathbf{v}}, \;\; \mathbf{x} = L\tilde{\mathbf{x}}, \;\; t = (L/V)\tilde{t}, \;\; (p/\rho) = V^2\tilde{p}$$

$$\partial_t \tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} = -\nabla\tilde{p} + \frac{1}{Re}\triangle\tilde{\mathbf{v}}$$

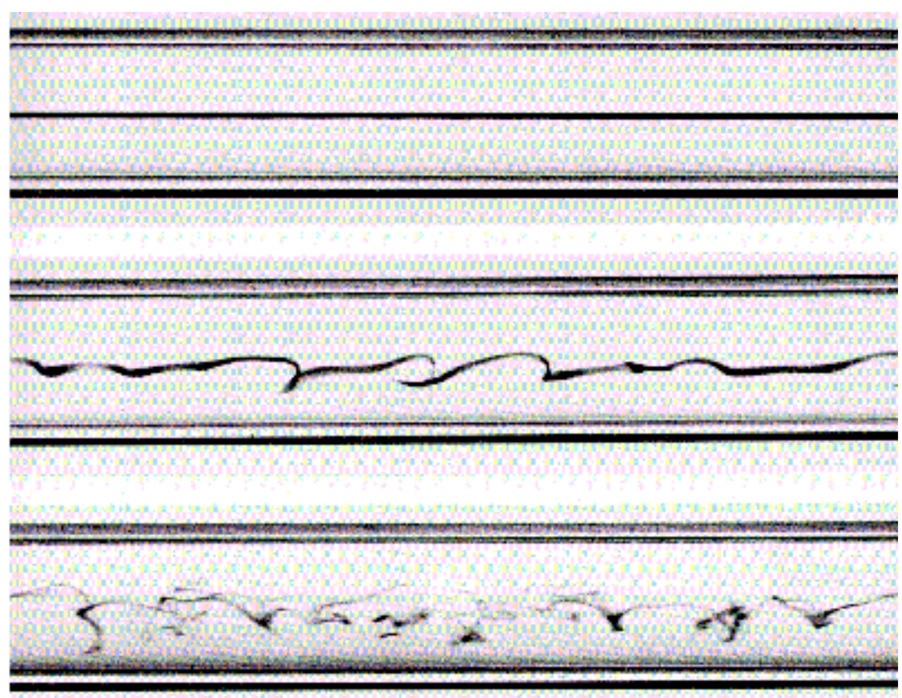
Similarity principle

$$\nabla \cdot \tilde{\mathbf{v}} = 0$$

$$\boxed{Re = \frac{VL}{\nu}}$$

Reynolds number

Flow speed increases



Conservation laws

Spatial average in a 3D periodic box: $\langle f \rangle \equiv \frac{1}{L^3} \int_{B_L} f(\boldsymbol{r}) d\boldsymbol{r}$

Vorticity: $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{v}.$

Kinetic energy

Enstrophy

Ideal invariants

$$E \equiv \left\langle \frac{1}{2} |\boldsymbol{v}|^2 \right\rangle, \quad \Omega \equiv \left\langle \frac{1}{2} |\boldsymbol{\omega}|^2 \right\rangle,$$

$$H \equiv \left\langle \frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{\omega} \right\rangle, \quad H_\omega \equiv \left\langle \frac{1}{2} \boldsymbol{\omega} \cdot \nabla \wedge \boldsymbol{\omega} \right\rangle$$

Helicity

Vortical helicity

$$\boxed{\frac{d}{dt} E = -2\nu \Omega, \qquad \frac{d}{dt} H = -2\nu H_\omega}$$

Dissipative anomaly

Energy dissipation rate: $\varepsilon \equiv -\dfrac{dE}{dt}$

$$\boxed{\lim_{\nu\to 0} \nu \int_V |\boldsymbol{\omega}|^2 \, \mathrm{d}\boldsymbol{x} \neq 0}$$

In turbulent flows, the energy dissipation rate *ε* has a finite limit as the viscosity ν tends to zero!

In this sense, the Euler and Navier-Stokes equations are fundamentally different!

Energy budget scale-by-scale

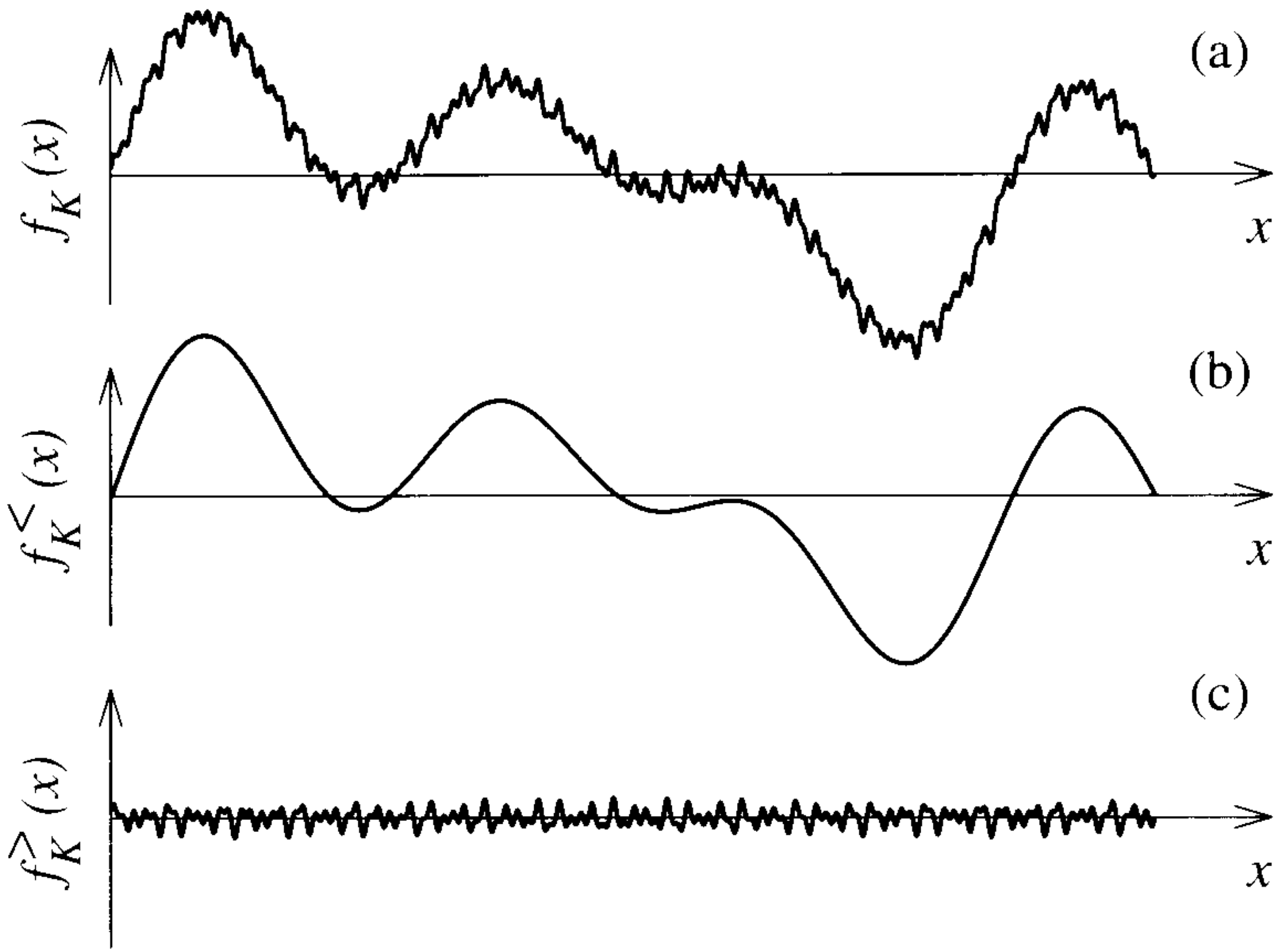
Low-/High-pass filter

$$f(\boldsymbol{r}) = f_K^{<}(\boldsymbol{r}) + f_K^{>}(\boldsymbol{r})$$

$$f(\boldsymbol{r}) = \sum_{k} \hat{f}_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$f_K^{<}(\boldsymbol{r}) \equiv \sum_{k \le K} \hat{f}_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$f_K^{>}(\boldsymbol{r}) \equiv \sum_{k > K} \hat{f}_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$



Energy budget scale-by-scale (cont'd)

Cumulative kinetic energy between wavenumbers 0 and K:

$$\mathcal{E}_K \equiv \frac{1}{2}\langle |\boldsymbol{v}_K^<|^2\rangle = \frac{1}{2}\sum_{k\leq K}|\hat{\boldsymbol{v}}_k|^2$$

Scale-by-scale energy budget equation:

$$\boxed{\partial_t \mathcal{E}_K + \Pi_K = -2\nu\Omega_K + \mathcal{F}_K}$$

Energy flux through wavenumber K:

$$\Pi_K \equiv \langle \boldsymbol{v}_K^< \cdot (\boldsymbol{v}_K^< \cdot \nabla \boldsymbol{v}_K^>)\rangle + \langle \boldsymbol{v}_K^< \cdot (\boldsymbol{v}_K^> \cdot \nabla \boldsymbol{v}_K^>)\rangle$$

Cumulative enstrophy:

$$\Omega_K \equiv \frac{1}{2}\langle |\omega_K^<|^2\rangle = \frac{1}{2}\sum_{k\leq K}k^2|\hat{\boldsymbol{v}}_k|^2$$

Cumulative energy injection:

$$\mathcal{F}_K \equiv \langle \boldsymbol{f}_K^< \cdot \boldsymbol{v}_K^<\rangle = \sum_{k\leq K}\hat{\boldsymbol{f}}_k \cdot \hat{\boldsymbol{v}}_{-k}$$

Spectral energy transfer

Energy spectrum: $$E(k) = \frac{\partial}{\partial_k}\frac{1}{2}\langle|\boldsymbol{v}_k^<|^2\rangle$$

Scale-by-scale energy transfer equation: $$\partial_t E(k) = T(k) + F(k) - 2\nu k^2 E(k)$$

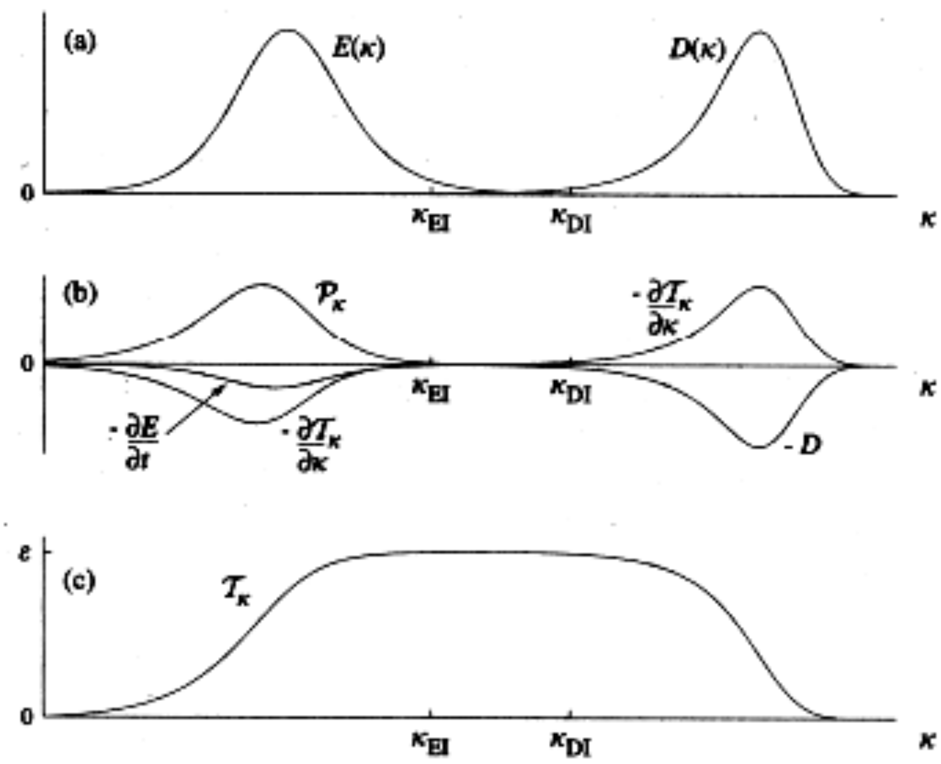
Net energy transfer spectrum: $$T(k) \equiv -\frac{\partial}{\partial_k}\Pi_k$$

Energy injection spectrum: $$F(k) = \frac{\partial}{\partial_k}\langle \boldsymbol{f}_k^< \cdot \boldsymbol{v}_k^< \rangle$$

Spectral energy transfer (cont'd)

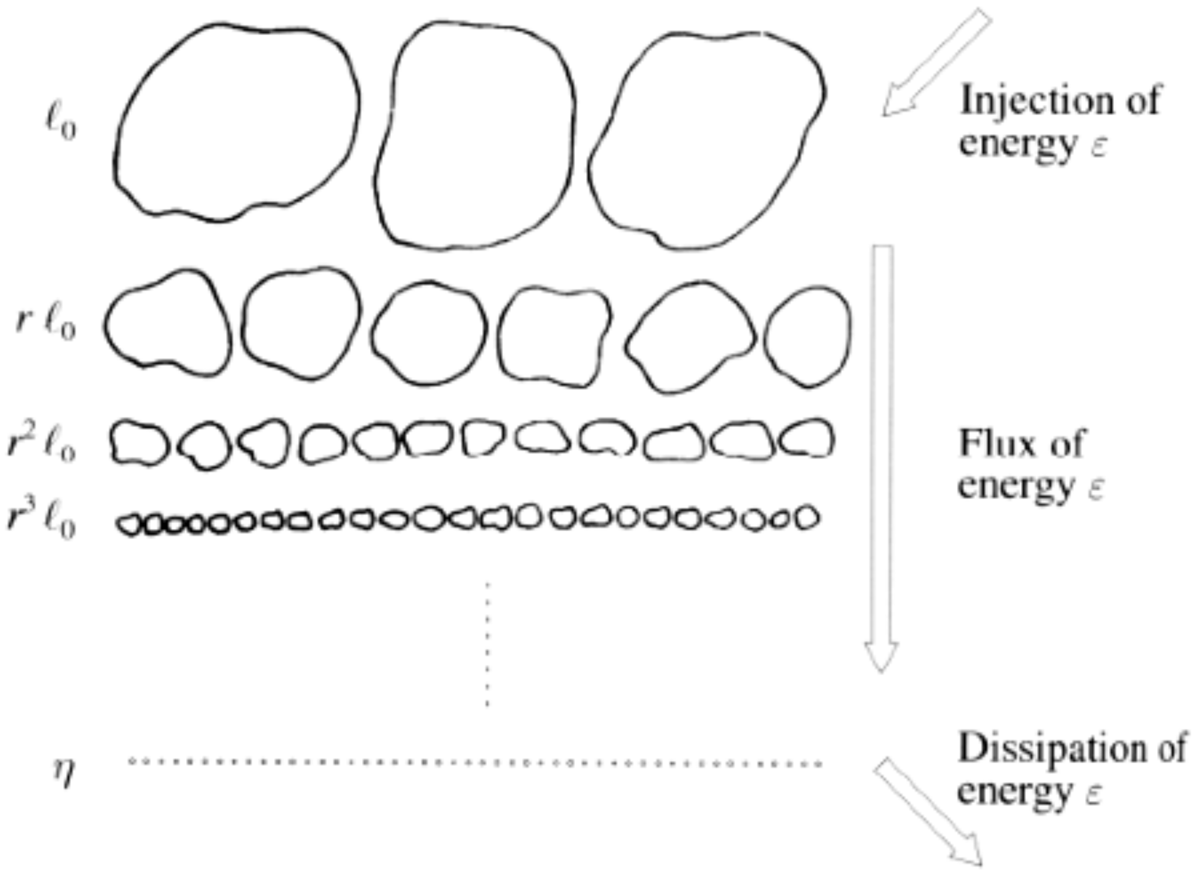
For homogeneous turbulence:

$$\frac{\partial}{\partial t}E(k,t) = \underbrace{\boxed{P_k(k,t)}}_{\text{Production}} \underbrace{\boxed{-\frac{\partial}{\partial k}T_k(k,t)}}_{\text{Spectral transfer}} \underbrace{\boxed{-2\nu k^2 E(k,t)}}_{\text{Dissipation}}$$



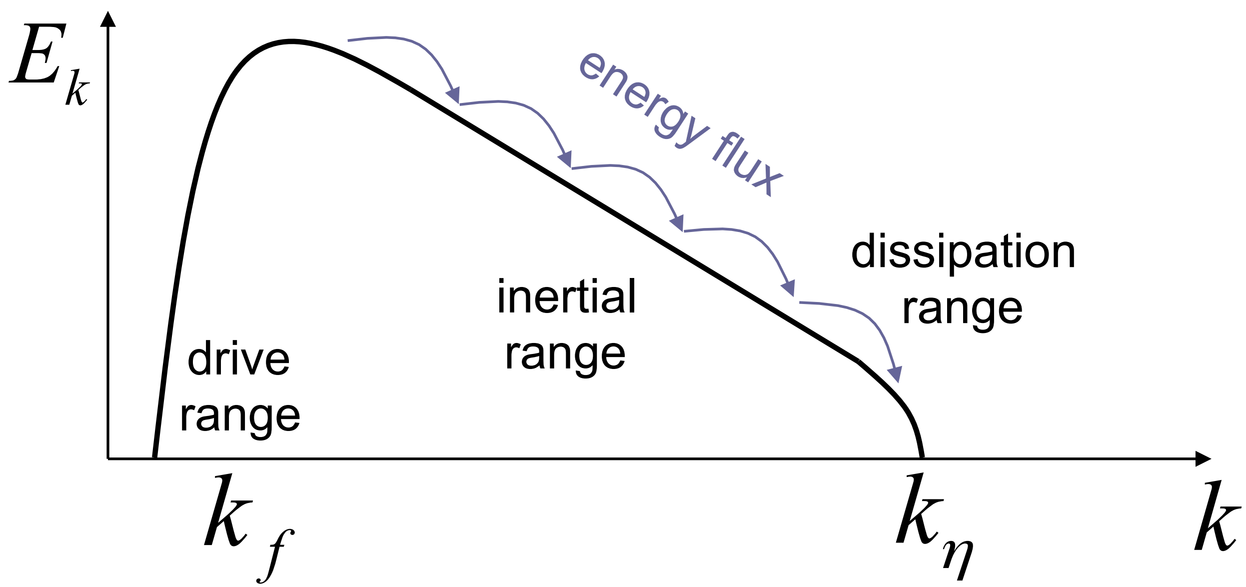
The Richardson cascade (real space)

„Big whorls have little whorls, little whorls have smaller whorls that feed on their velocity, and so on to viscosity"



The Richardson cascade (Fourier space)

Turbulence as a **local cascade in wave number space**…



$$\frac{k_\eta}{k_f} \sim \mathrm{Re}^{3/4}$$

Computational effort $\sim \mathrm{Re}^3$

Much turbulence research addresses the **cascade** problem.
(Important note: In this context, think of an Autobahn, not of a waterfall…)

Kolmogorov’s theory from 1941

K41 is based merely on intuition and dimensional analysis – it is *not* derived rigorously from the Navier-Stokes equation

Key assumptions:

- Scale invariance – like, e.g., in critical phenomena
- Central quantity: energy flux ε

$$E = \frac{1}{2V}\int v^2\, d^3x = \int_0^\infty E(k)\, dk$$

$$\boxed{E(k) = C\, \varepsilon^{2/3}\, k^{-5/3}}$$

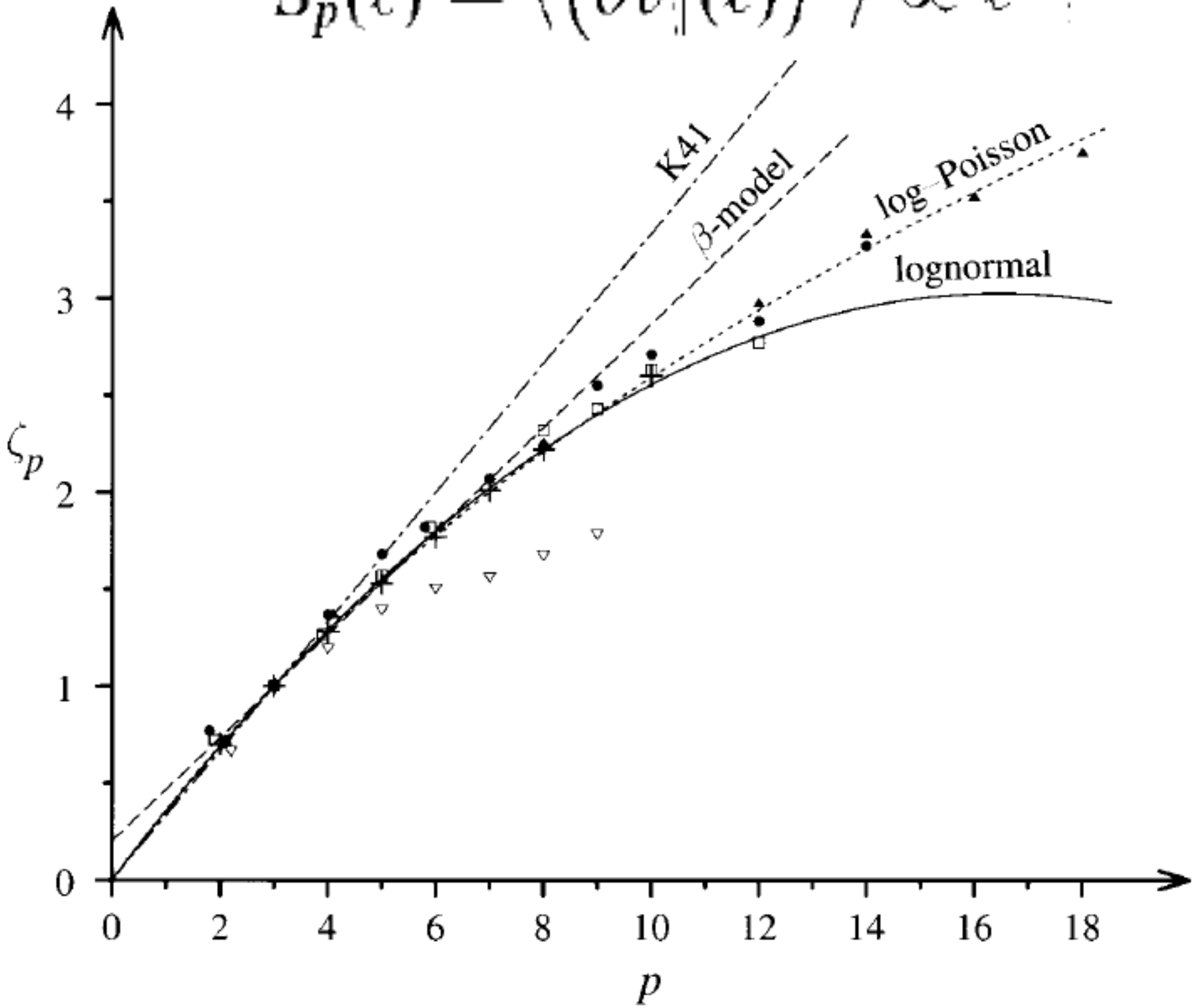
Quantity	Dimension
Wave number	1/length
Energy per unit mass	$\text{length}^2/\text{time}^2$
Energy spectrum $\mathcal{E}(k)$	$\text{length}^3/\text{time}^2$
Energy flux ε	energy/time $\sim \text{length}^2/\text{time}^3$

This is the most famous turbulence result: the “-5/3” law.

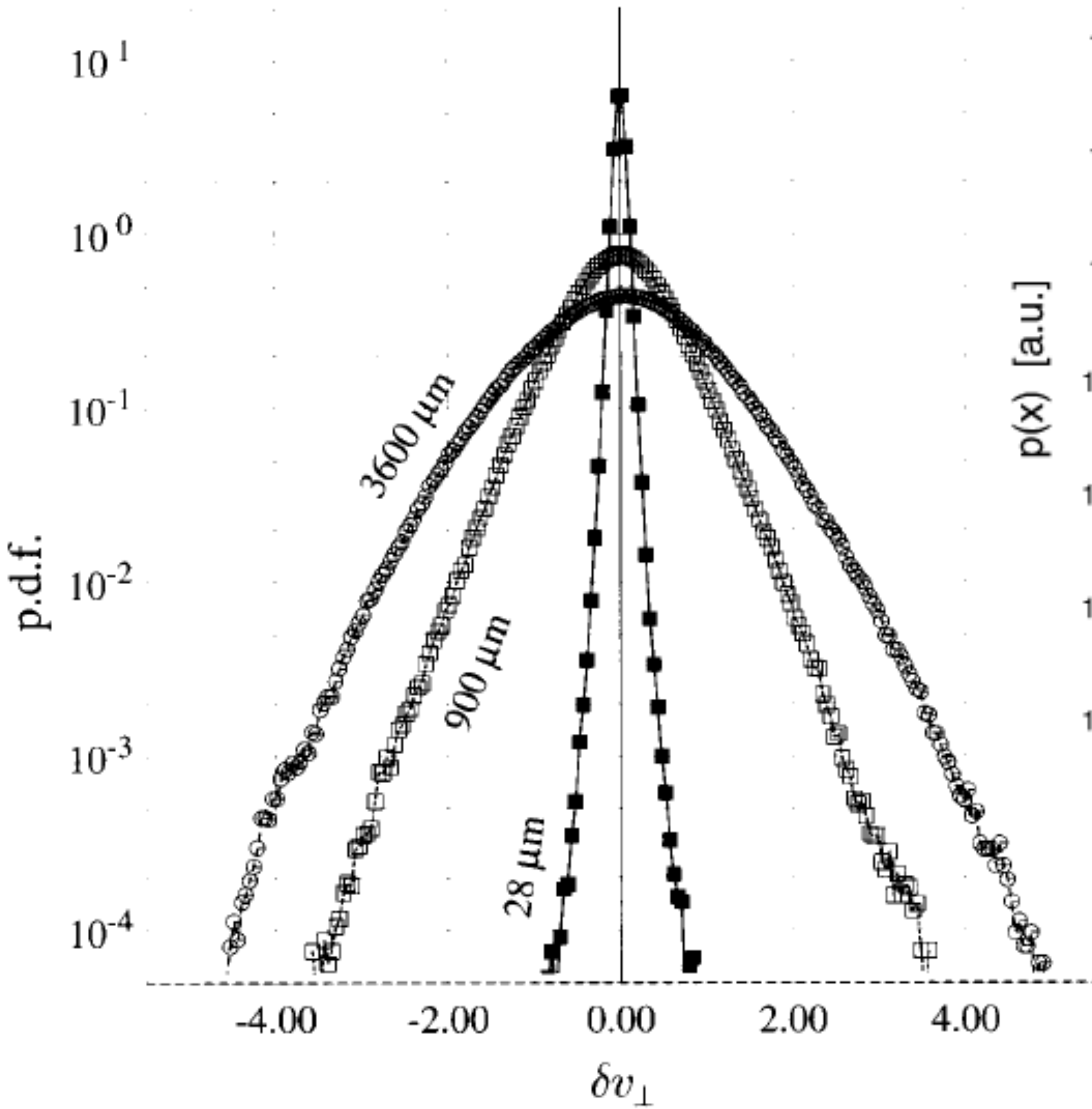
However, K41 is fundamentally wrong: scale invariance is broken (anomaly)!

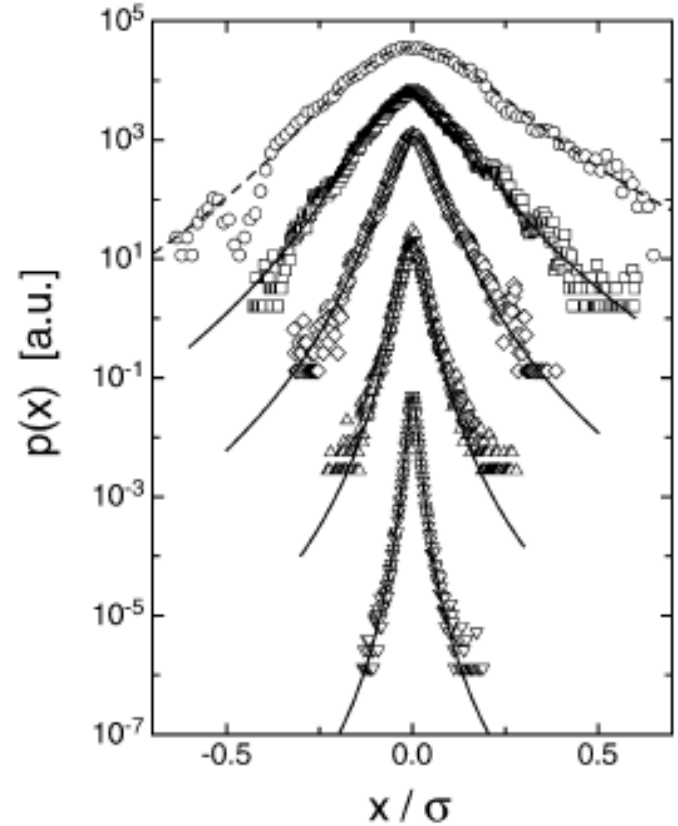
Intermittency & structure functions

$$S_p(\ell) = \langle (\delta v_{\|}(\ell))^p \rangle \propto \ell^{\zeta_p}$$



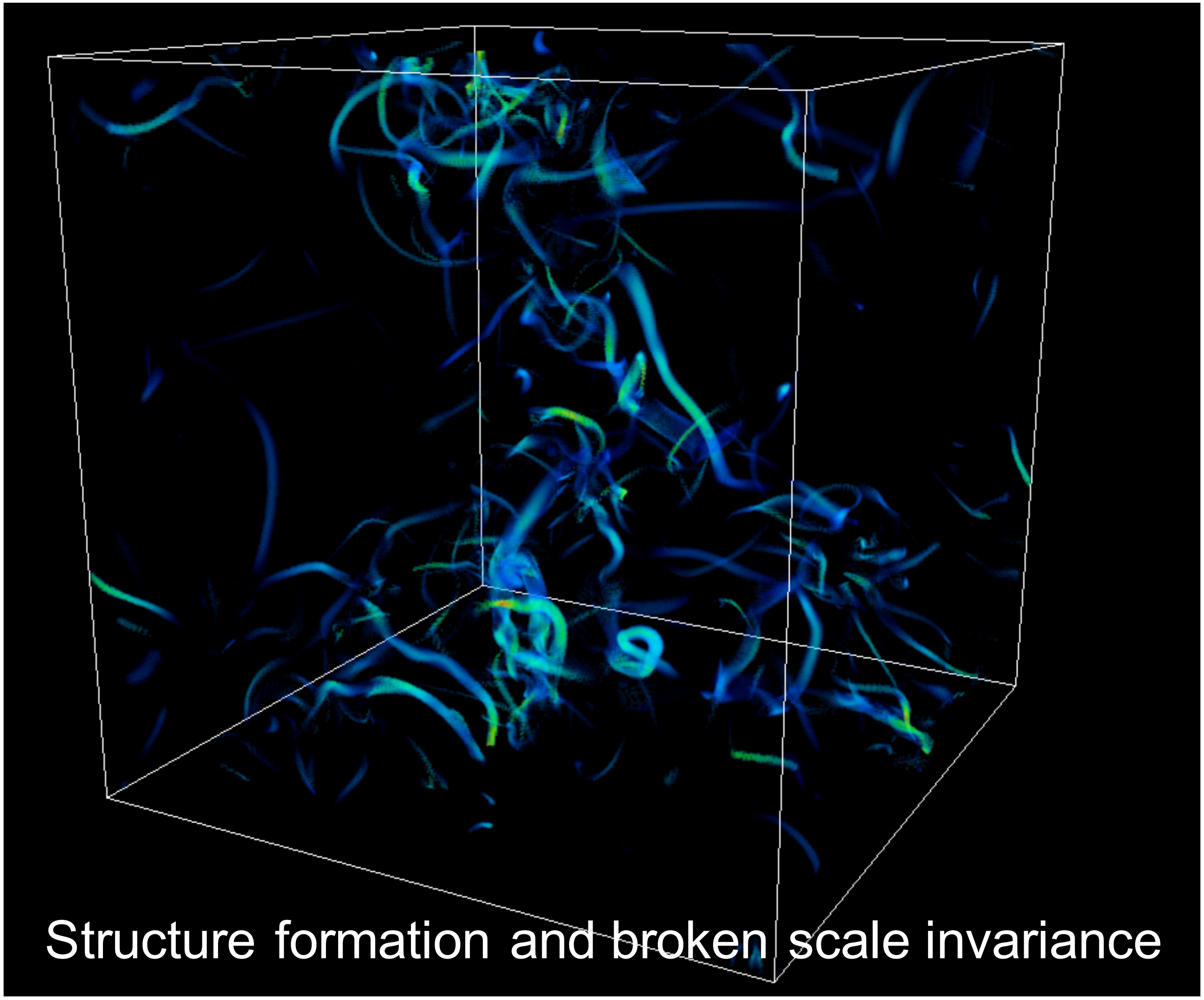
Intermittency & non-Gaussian pdf's





Renner et al. 2001
financial data
(exchange rates)

Direct numerical simulations



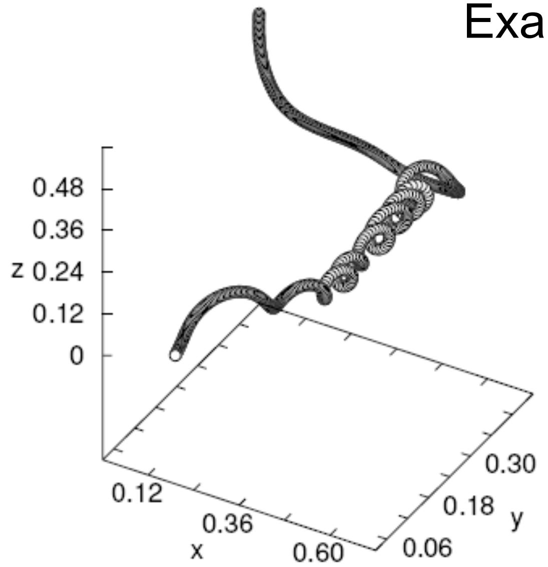
Structure formation and broken scale invariance

Wilczek *et al.* 2008

Key open issues: Inertial range

- Is the inertial range physics universal (for Re $\rightarrow \infty$)?
- If so, can one derive a rigorous IR theory from the NSE?
- How should one, in general, handle the interplay between randomness and coherence? Key issue: **Intermittency!**

Biferale et al. 2005



Example: Trapping of tracers in vortex filaments

Note:

The observed deviations from self-similarity can be reproduced qualitatively by relatively simple vortex models.

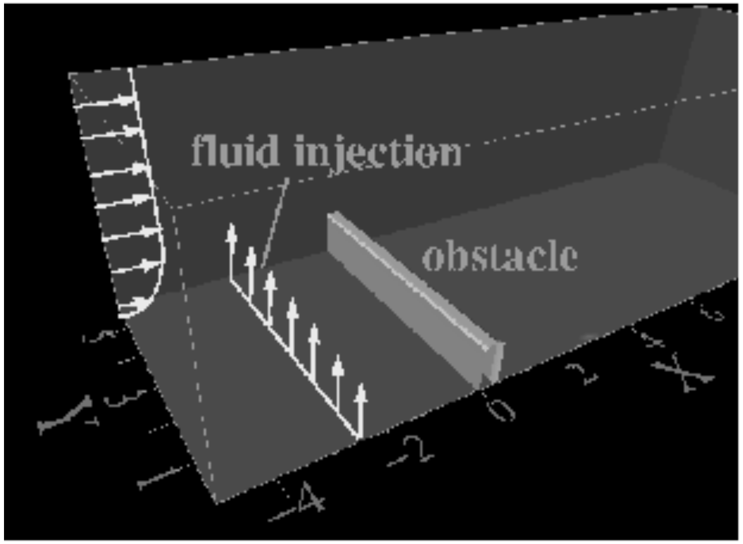
Wilczek, Jenko, and Friedrichs 2008

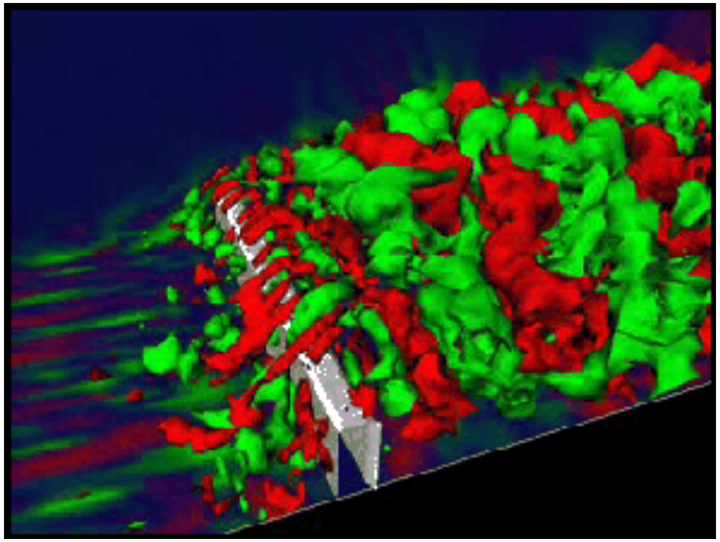
Key open issues: Drive range

- Often, one is interested mainly in the *large* scales. Here, one encounters an interesting interplay between linear (drive) and nonlinear (damping) physics. – Is it possible to remove the small scales?

- Candidates: LES, dynamical systems approach etc.

Orellano and Wengle 2001

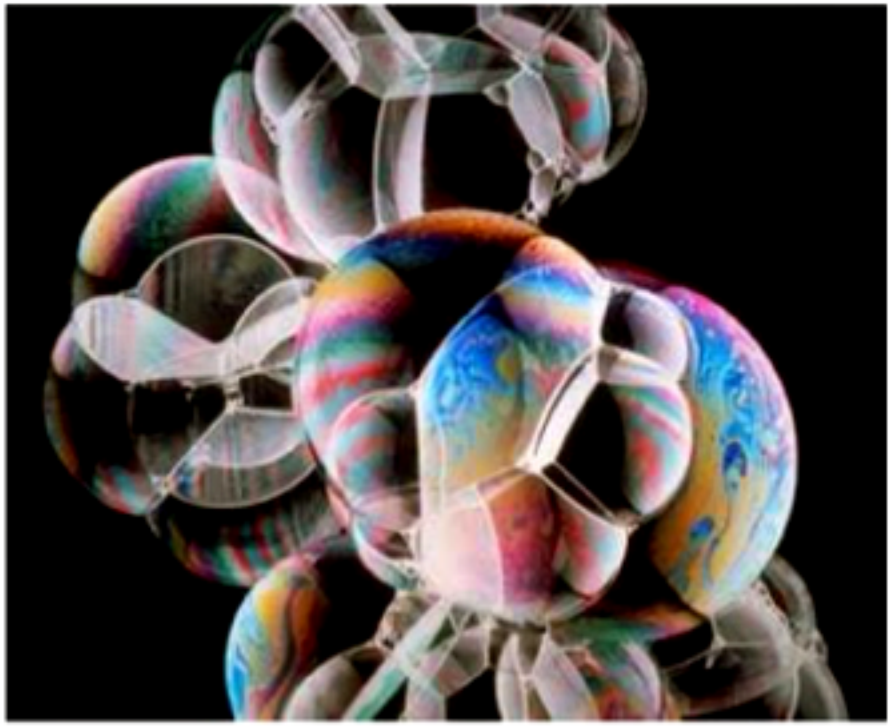




On the physics of 2D fluid turbulence

2D turbulence?

- strictly speaking, there are no two-dimensional flows in nature
- approximately 2D: soap films, stratified fluids, geophysical flows, magnetized plasmas



Basic equations: Vorticity

Taking the curl of the NS equations and discarding the zero x and y components of the equation gives

$$\boxed{\frac{D}{Dt}\omega = \left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\omega = g + \nu\nabla^2\omega}$$

for the vorticity $\omega = \left(\vec{\nabla}\times\vec{v}\right)\cdot\hat{z}$.

If $g = (\vec{\nabla}\times\vec{f}_{ext})\cdot\hat{z} = 0$ and $\nu = 0$ (Euler equation), we have $\frac{D}{Dt}\omega = 0$
$\longrightarrow$ vorticity is conserved

Energy and enstrophy

mean energy $E = \langle \frac{1}{2} u^2 \rangle$ $\qquad$ $\frac{dE}{dt} = -2\nu Z$

enstrophy $Z = \langle \frac{1}{2} \omega^2 \rangle$ $\qquad$ $\frac{dZ}{dt} = -2\nu \langle (\nabla \omega)^2 \rangle$

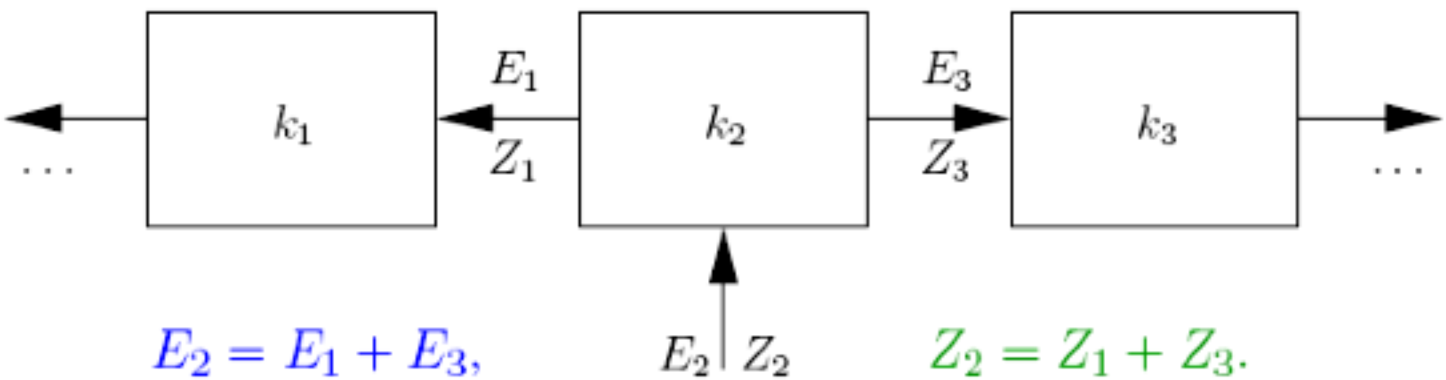
(mean square vorticity)

- in 2D with curl free forcing, energy and enstrophy can only decrease with time, they are conserved in the inviscid case ($\nu = 0$)
- for $\nu \to 0$ we get $\frac{dE}{dt} \to 0$, energy is a "robust" invariant in 2D
- $\frac{dZ}{dt}$ does not necessarily go to zero for $\nu \to 0$, enstrophy is a "fragile invariant" (dissipation anomaly!)

Cascades

In spectral space, the expressions for energy and enstrophy read

$$E = \int E(k,t)dk$$
$$Z = \int k^2 E(k,t)dk$$



$E_2 = E_1 + E_3,$ $\quad$ $Z_2 = Z_1 + Z_3.$

Energy and enstrophy conservation for three Fourier modes k_1, $k_2 = 2k_1$, $k_3 = 3k_1$

Dual cascade

$$\begin{aligned} \delta E_1 + \delta E_2 + \delta E_3 &= 0 \\ k_1^2 \delta E_1 + k_2^2 \delta E_2 + k_3^2 \delta E_3 &= 0 \end{aligned}$$

with $\delta E_i = E(k_i, t_2) - E(k_i, t_1)$. Combining the equations gives

$$\begin{aligned} \delta E_1 &= -\frac{5}{8} \delta E_2 \\ k_1^2 \delta E_1 &= -\frac{5}{32} k_2^2 \delta E_2 \end{aligned} \qquad \begin{aligned} \delta E_3 &= -\frac{3}{8} \delta E_2 \\ k_3^2 \delta E_3 &= -\frac{27}{32} k_2^2 \delta E_2 \end{aligned}$$

$\rightarrow$ enstrophy goes to higher k (direct enstrophy cascade),

energy goes to lower k (inverse energy cascade)

Kraichnan-Batchelor-Leith theory

- inertial range of the energy cascade: for $k_E \ll k \ll k_i$, the energy spectrum can only depend on ϵ.
  Dimensional analysis:

  $$k = [L]^{-1};\ E(k) = [L]^3[T]^{-2};\ \epsilon = [L]^2[T]^{-3}\ \rightarrow\ E(k) = C\epsilon^{2/3}k^{-5/3}$$

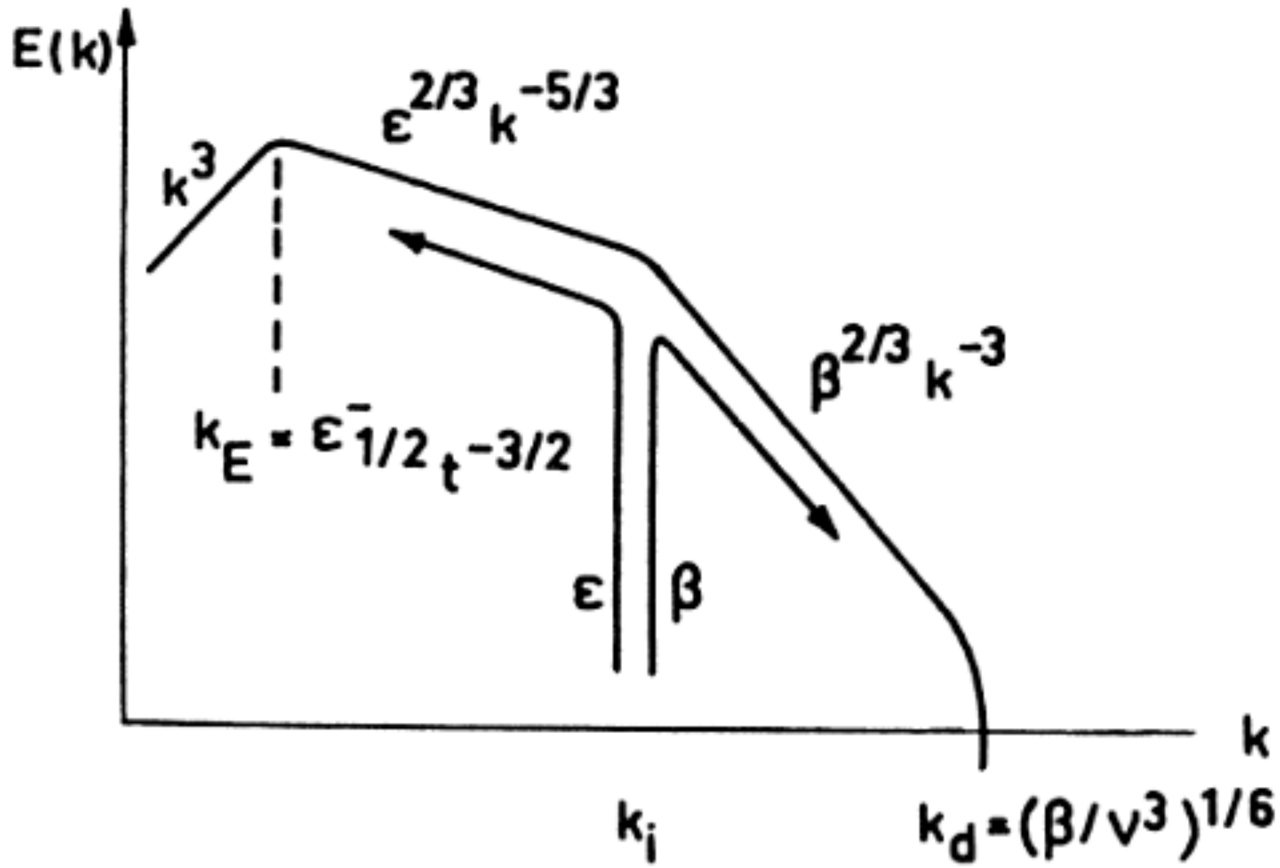
- inertial range of the enstrophy cascade ($k_i \ll k \ll k_d$): the energy spectrum can only depend on β.
  Dimensional analysis:

  $$k = [L]^{-1};\ E(k) = [L]^3[T]^{-2};\ \beta = [T]^{-3}\ \rightarrow\ E(k) = C'\beta^{2/3}k^{-3}$$

  C, C' constant and dimensionless.

- zero enstrophy transfer in the energy inertial range, zero energy transfer in the enstrophy inertial range

Inertial ranges in 2D turbulence



The Hasegawa-Mima model of drift-wave turbulence (1977)

Charney-Hasegawa-Mima equation

Hasegawa & Mima, PRL 1977

In a certain limiting case (in particular: cold ions), gyrokinetics leads to the CHM equation which is closely related to the 2D NS equation; used in geophysics already since 1948...

$$\frac{d}{dt}(\phi - \nabla^2 \phi - x) = 0$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} - \frac{\partial \phi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial y}$$

One-field model (for the electrostatic potential); no linear drive/damping

J. G. Charney

Basic assumptions

Homogeneous magnetic field: $\mathbf{B} = B\,\hat{\mathbf{z}}$

Electrostatic fluctuations: $\mathbf{E} = -\nabla\tilde{\phi}$

Slowly varying background density: $\nabla n_0 = -(n_0/L_n)\,\hat{\mathbf{x}}$

„Cold" ions: $T_i \ll T_e$

„Adiabatic" electrons!

Perpendicular ion dynamics

Force balance: $$m_i n_i \left(\frac{\partial}{\partial t} + \mathbf{v}_\perp \cdot \nabla \right) \mathbf{v}_\perp = e n_i \left(\mathbf{E} + \mathbf{v}_\perp \times \mathbf{B} \right)$$

Expand in orders of ω/Ω_i: $$\mathbf{v}_\perp = \mathbf{v}_\perp^{(0)} + \mathbf{v}_\perp^{(1)} + \ldots$$

0th order: $$\boxed{\mathbf{v}_\perp^{(0)} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \equiv \mathbf{v}_E}$$ ExB drift

1st order: $$m_i n_i \left(\frac{\partial}{\partial t} + \mathbf{v}_\perp^{(0)} \cdot \nabla \right) \mathbf{v}_\perp^{(0)} = e n_i \mathbf{v}_\perp^{(1)} \times \mathbf{B}$$

$$\boxed{\mathbf{v}_\perp^{(1)} = \frac{1}{\Omega_i B} \mathbf{B} \times \left(\frac{\partial}{\partial t} + \mathbf{v}_\perp^{(0)} \cdot \nabla \right) \mathbf{v}_\perp^{(0)} \equiv \mathbf{v}_p}$$ polari-zation drift

Perpendicular ion dynamics (cont'd)

Ion continuity equation:

$$\partial_t \tilde{n} + \mathbf{v}_\perp \cdot \nabla n_0 + \mathbf{v}_\perp \cdot \nabla \tilde{n} + n_0 \nabla \cdot \mathbf{v}_\perp + \tilde{n} \nabla \cdot \mathbf{v}_\perp = 0$$

Low-order expansion: $\mathbf{v}_\perp \longrightarrow \mathbf{v}_\perp^{(0)}, \quad \nabla \cdot \mathbf{v}_\perp \longrightarrow \nabla \cdot \mathbf{v}_\perp^{(1)}$

$$\nabla \cdot \mathbf{v}_\perp^{(0)} = 0, \quad \nabla \cdot \mathbf{v}_\perp^{(1)} = \frac{1}{\Omega_i B} \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \nabla_\perp^2 \tilde{\phi}$$

Hasegawa-Mima equation (Charney equation) in 2D:

$$\boxed{\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \left[\left(\frac{e\tilde{\phi}}{T_e} \right) - \rho_s^2 \nabla_\perp^2 \left(\frac{e\tilde{\phi}}{T_e} \right) - \left(\frac{x}{L_n} \right) \right] = 0}$$

Non-dimensionalized HME

Ion sound radius / speed: $$\rho_s \equiv \frac{c_s}{\Omega_i}, \quad c_s^2 \equiv \frac{T_e}{m_i}$$

Normalization:

$$\frac{e\tilde{\phi}}{T_e}\frac{L_n}{\rho_s} \longrightarrow \phi, \quad \frac{x}{\rho_s} \longrightarrow x, \quad \frac{y}{\rho_s} \longrightarrow y, \quad \frac{c_s t}{L_n} \longrightarrow t$$

HME: $$\boxed{\frac{d}{dt}\left(\phi - \nabla_\perp^2 \phi - x\right) = 0}$$ (similar to 2D NSE)

$$\frac{d}{dt} \equiv -(\nabla_\perp \phi \times \hat{\mathbf{z}}) \cdot \nabla_\perp = \frac{\partial}{\partial t} - \frac{\partial \phi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial y}$$

Similarity between HME and NSE

2D incompressible flow; introduce stream function:

$$\mathbf{u} = \hat{z} \times \nabla\psi \quad \text{where} \quad \psi = \psi(x, y) \Rightarrow \nabla \cdot \mathbf{u} = 0$$

2D NSE:

vorticity

$$\boxed{\left(\frac{\partial}{\partial t} - \nabla\psi \times \hat{z} \cdot \nabla\right) \nabla^2\psi = 0}$$

This corresponds to the HME in the high-k limit (and without a background density gradient)

Electron drift waves

Linearize HME and use Ansatz: $\phi \propto \exp(ik_x x + ik_y y - i\omega t)$

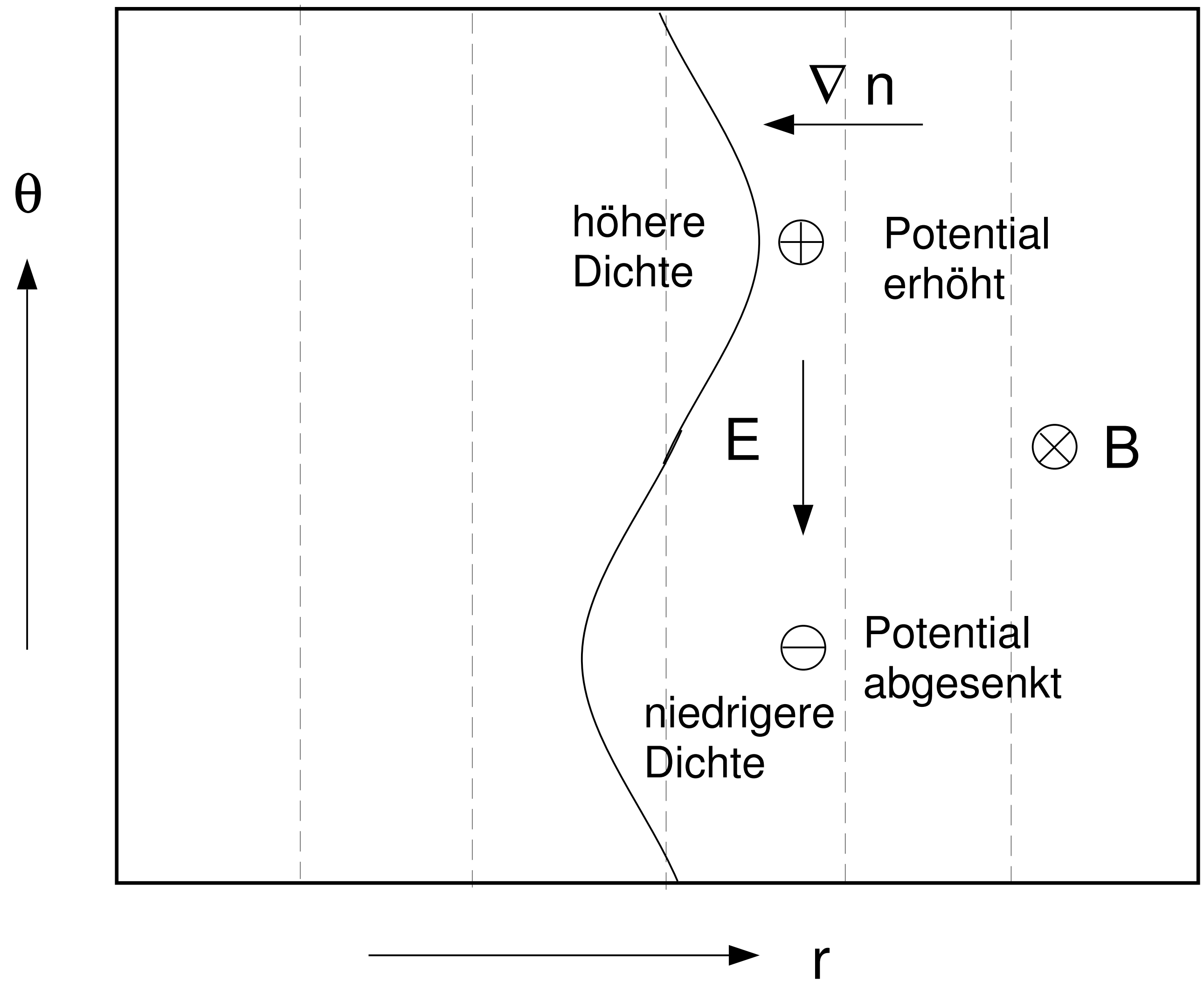
Linear dispersion relation: $\boxed{\omega_D = \frac{k_y \rho_s}{1 + k_\perp^2 \rho_s^2} \frac{c_s}{L_n}}$ electron drift waves

Waves drift in the electron diamagnetic (y) direction:

$$v_{\mathrm{ph}} \equiv \frac{\omega_D}{k_y} \approx v_{\mathrm{g}} \equiv \frac{\partial \omega_D}{\partial k_y} \approx v_{de} \equiv \frac{\rho_s c_s}{L_n}$$

Re-interpretation: $D^\perp \sim \frac{\omega_D}{k_\perp^2} \sim \frac{1}{k_\perp \rho_s} \frac{\rho_s^2 c_s}{L_n} \sim (3-10) \frac{\rho_s^2 c_s}{L_n}$

Electron drift waves (cont'd)



Cascades and wavenumber spectra

Cmp. measurements of density fluctuation spectra via microwave and laser scattering since ~1976

Ideal invariants:

$$W = \int dV \, \frac{\phi^2 + (\nabla\phi)^2}{2}$$

generalised energy

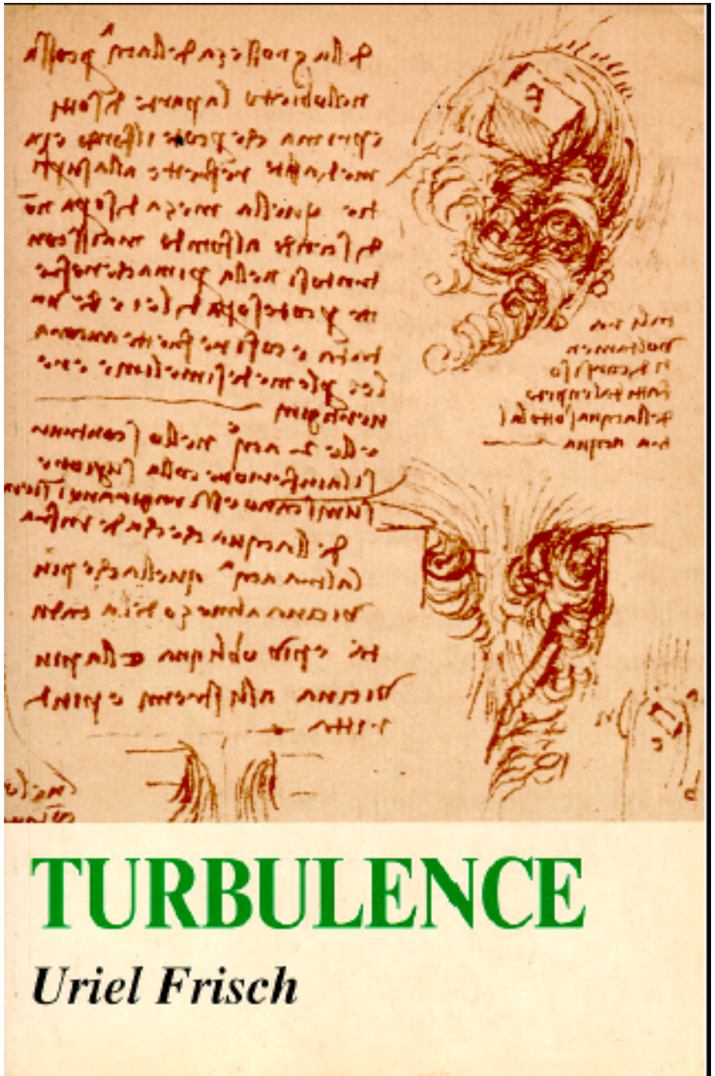
$$U = \int dV \, \frac{(\nabla\phi)^2 + (\nabla^2\phi)^2}{2}$$

generalised enstrophy

	inverse energy cascade	direct enstrophy cascade
k>>1	$W(k) = c\epsilon^{-2/3} k^{-5/3}$	$W(k) = c'\eta^{2/3} k^{-3}$
k<<1	$\sim \epsilon^{2/3} k^{-11/3}$	$\sim \eta^{2/3} k^{-5}$

Turbulence: Further reading

Recommended reading



Also:

P. A. Davidson
Turbulence

S. B. Pope
Turbulent flows

Some literature on 2D turbulence

General 2D turbulence:

- P.A. Davidson, Turbulence, Oxford University Press (2004)
- M. Lesieur, Turbulence in Fluids, Kluwer (1997)
- U. Frisch, Turbulence, Cambridge University Press (1995)
- P. Tabelling, Two-dimensional turbulence: a physicist approach, Phys. Rep. 362, 1-62 (2002)

Cascade classics

- Kraichnan, Inertial Ranges in Two-Dimensional Turbulence, Phys. Fluids 10, 1417 (1967)
- Leith, Diffusion Approximation for Two-Dimensional Turbulence, Phys. Fluids 11, 1612 (1968)
- Batchelor, Computation of the Energy Spectrum in Homogenous Two-Dimensional Turbulence, Phys. Fluids 12, II-233 (1969)

Fundamental literature on the HME

Stationary spectrum of strong turbulence in magnetized nonuniform plasma

A. Hasegawa and K. Mima, Phys. Rev. Lett. **39**, 205 (1977)

See also: Phys. Fluids **21**, 87 (1978); **22**, 2122 (1979)

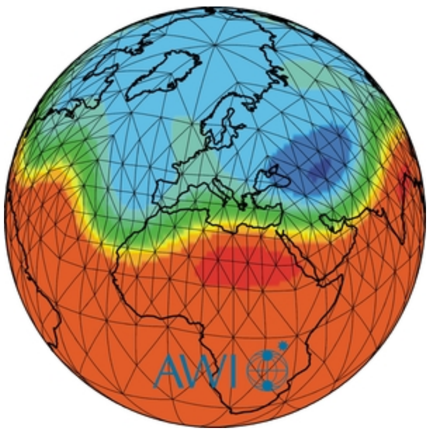
Quasi-two-dimensional dynamics of plasmas and fluids

W. Horton and A. Hasegawa, Chaos **4**, 227 (1994)

A. Hasegawa (*1934): Maxwell Prize (APS) 2000; Alfvén Prize (EPS) 2011

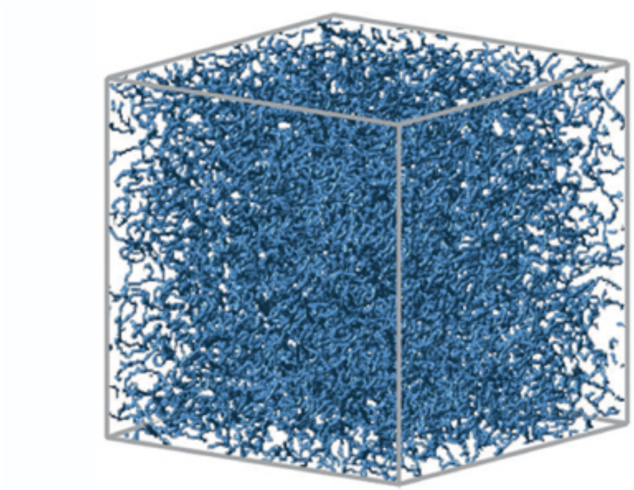
Plasma turbulence: The context

Strong wave turbulence



Turbulence in planetary atmospheres: Rossby waves

Turbulence in oceans:
Water surface waves



Turbulence in quantum liquids:
Kelvin waves on vortex filaments