Accretion discs
Les Houches school 2017, lecture notes

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1 Accretion disk description

1.1 Thin Disk equilibrium

An accretion disc is typically made of gaz (and possibly dust) orbiting a central object (young star, white dwarf, neutron star of black hole) of mass M. In this lecture, we assume that the gravity of the orbiting gas onto itself (self-gravity) is negligible. This is however not necessarily in very massive discs (young protostellar discs, broad line regions in AGNs). Under these assumptions, the gravitational potential is simply that of the central object and the equilibrium may simply be written

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial R} - \partial_R \psi + \Omega^2 R \tag{1}$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} - \partial_z \psi, \tag{2}$$

where (R, z) are cylindrical coordinates and Ω is the angular velocity of the flow, which we assume only depends on R and $\psi = -GM/(R^2 + z^2)^{1/2}$ is the cylindrical potential. A useful quantity will be the Keplerian frequency which corresponds to the orbital frequency of a test a particle on a circular orbit at radius R :

$$\Omega_K(R) = \sqrt{\frac{GM}{R^3}} \tag{3}$$

In order to simplify the computation, we assume that the disc is locally isothermal : $T(R)$. Under these assumptions, the sound speed may be written

$$c_s \equiv \sqrt{\frac{P}{\rho}} \tag{4}$$

$$= \sqrt{\frac{kT}{\mu}} \tag{5}$$

where k is Boltzmann's constant and μ is the mean molecular mass. Since the disc is locally isothermal, c_s only depends on R, as the temperature does.

We first start with the vertical equilibrium which we consider close to the disc midplane ($z \ll R$) since we assume the disc is thin :

$$c_s^2 \partial_z \log \rho = -\frac{GMz}{(R^2 + z^2)^{3/2}} \tag{6}$$

$$\simeq z\Omega_K^2 + O(z^2) \tag{7}$$

where we have assumed $z \ll R$. We deduce from this the vertical density profile

$$\rho = \rho_0(r) \exp\Big(-\frac{z^2}{2H^2}\Big) \tag{8}$$

where we have defined the disc scale height

$$H \equiv c_s/\Omega_K. \tag{9}$$

Hence the thin disc approximation $H \ll R$ implies that the disc is cold, or in other words that $c_s \ll R\Omega_K$. This is is the case for many discs in astrophysical systems (protostellar disc, cataclysmic variable, outer part of X-ray binaries).

In the radial direction we first have to compare the radial pressure gradient to the gravitational potential

$$0 = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial R}}_{\sim c_s^2/R} - \underbrace{\partial_R \psi}_{\sim \Omega_K^2 R} + \Omega^2 R \tag{10}$$

$$\tag{11}$$

Hence, the pressure gradient is $(H/R)^2$ smaller than the gravitational potential and can be neglected in the thin disc approximation. This means that the disc is *to a very good approximation* a keplerian disc $\Omega = \Omega_K$.

1.2 Accretion theory in a thin disk

1.2.1 General conservation equations

The accretion of mass in astrophysical discs is easily describe by the equation of mass and angular momentum conservation. Let us start with mass conservation :

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{u} = 0 \tag{12}$$

which we average vertically defining $\overline{\,\cdot\,}$ as

$$\overline{Q} = \int d\phi \int_{z=-h}^{z=+h} dz\, Q. \tag{13}$$

which leads to :

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R} R\overline{\rho u_r} + \Big[\rho v_z\Big]_{z=-h}^{+h} = 0 \tag{14}$$

where $\Sigma \equiv= \overline{\rho}$ is the surface density of the gas.

The equation of angular momentum conservation may be written as

$$\frac{\partial(\rho R u_\phi)}{\partial t} + \boldsymbol{\nabla\cdot}\left[\rho R u_\phi \boldsymbol{u} - RB_\phi \boldsymbol{B} + \left(P + \frac{B^2}{2}\right)\boldsymbol{e_\phi}\right] = 0 \tag{15}$$

We separate u_ϕ into deviations plus mean flow :

$$u_r = v_r \tag{16}$$
$$u_\phi = \Omega_K R + v_\phi \tag{17}$$
$$u_z = v_z \tag{18}$$

and we assume these deviation are small compared to the keplerian flow ($v \ll \Omega R$). Angular momentum reads

$$\Omega_K R^2 \frac{\partial\rho}{\partial t} + \frac{\partial(\rho R v_\phi)}{\partial t} + \boldsymbol{\nabla\cdot}\left[\rho R^2 \Omega_K \boldsymbol{v} + \rho R v_\phi \boldsymbol{v} - RB_\phi \boldsymbol{B} + \left(P + \frac{B^2}{2}\right)\boldsymbol{e_\phi}\right] = 0 \tag{19}$$

We then apply the averaging procedure (13) defined above to get

$$\Omega R^2 \frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}R\left(R^2\Omega\overline{\rho v_r} + R\overline{\rho v_\phi v_r} - R\overline{B_\phi B_r}\right) \tag{20}$$

$$+\left[R^2\Omega\rho v_z + R\rho v_\phi v_z - RB_\phi B_z\right]_{z=-h}^{+h} = 0 \tag{21}$$

Angular momentum conservation can then be combined with mass conservation (14) to get a final equation expressing the mass accretion rate $\overline{\rho v_r}$ as a function of the radial and surface stresses

$$\overline{\rho v_r}\frac{\partial}{\partial R}\Omega_K R^2 + \frac{1}{R}\frac{\partial}{\partial R}R^2\underbrace{[\overline{\rho v_\phi v_r} - \overline{B_\phi B_r}]}_{\text{Radial stress}} + \underbrace{\left[R\rho v_\phi v_z - RB_\phi B_z\right]_{z=-h}^{+h}}_{\text{Surface stress}} = 0. \tag{22}$$

This demonstrates the close relationship between the accretion rate and the transport of angular momentum by the stresses. Angular momentum can be transported outward in the disc by the radial stress, or evacuated from the disc by a torque applied at the disc surface, as for example when a magnetised wind is present.

1.2.2 Viscous theory

The viscous theory assumes no wind is present at the disc surface. In order to solve the long-term evolution of the disc, one needs to express the radial stress

$$W_{r\phi} = \overline{\rho v_\phi v_r} - \overline{B_\phi B_r} \tag{23}$$

as a function of vertically average quantities such as Σ, $\overline{P}$, etc. Historically, and based on a purely dimensional argument (Shakura & Sunyaev 1973), it is usually assumed that

$$W_{r\phi} = \alpha\overline{P}, \tag{24}$$

where α is a dimensionless constant. Physically, it can however be justified a bit more : let us consider turbulent fluctuations v in a thin disc. The fluctuation are confined in the disc thickness H and they are excited at the Keplerian frequency Ω_K. Hence, we expect $v \sim H\Omega_K$ and therefore $W_{r\phi} \sim \overline{\rho v^2} \sim \overline{\rho H^2 \Omega_K^2}$. Using (9), one gets $W_{r\phi} \sim \overline{\rho c_s^2} \sim \overline{P}$. Hence, thanks to the vertical equilibrium of a thin disc, Shakura & Sunyaev 1973 prescription is fully justified!

This prescription may be considered as a viscous theory. Indeed, $W_{r\phi} = \alpha \overline{P} = \alpha \Sigma c_s H \Omega_K$. Since Ω_K is the local shear rate of the flow, the stress in the angular momentum conservation equation shows up as a viscous stress $\nu_t \Sigma \Omega$ with a turbulent viscosity coefficient $\nu_t = \alpha c_s H$.

Plugging this prescription in (22) leads to

$$\overline{\rho v_r} = -\frac{1}{R\partial_R(\Omega_K R^2)}\frac{\partial}{\partial R}R^2 \alpha c_s^2 \Sigma \tag{25}$$

Which can be combined with mass conservation to get an equation on Σ :

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left[\frac{1}{\partial_R(\Omega_K R^2)}\frac{\partial}{\partial R}R^2 \alpha c_s^2 \Sigma\right] \tag{26}$$

Which essentially constitutes a diffusion equation for the surface density. The diffusion timescale associated to accretion can be estimated using the fact that $c_s = \Omega_K H$. One finds

$$\tau_{\mathrm{visc}} \sim \alpha\Omega\left(\frac{H}{R}\right)^2 \ll \Omega_K \tag{27}$$

Accretion therefore occurs on timescales much longer than the orbital timescale in thin discs. This is usually a problem for simulations trying to capture the phenomenon of accretion. However, it allows us to separate accretion from dynamics occuring at the local orbital frequency, by stating that accretion is essentially inexistent on this timescale.

1.3 Estimated accretion time

2 Accretion disc stability

2.1 Local effective gravity

The Hill's approximation is essentially a local approximation designed to capture the physics of a patch of gas or particles orbiting a central object of any sort.

Let us consider particles (stars, fluid particles, dust), on a exactly circular orbit in a gravitational potential ψ. Each particle is at a radius R from the center of mass O, and orbits the center with an angular velocity $\Omega_K(R)$.

The radial equilibrium for the particles may be simply written

$$\Omega^2 R - \partial_R \psi = 0. \tag{28}$$

Let us consider a fiducial radius R_0 and a rotating frame $\mathcal{R}'$ attached to the circular orbit at R_0. The $\mathcal{R}'$ is rotating at $\Omega_0 \equiv \Omega_K(R_0)$ and we attach cartesian coordinates (x, y, z) so that x is align with the radius and y with the azimuth (Fig. 1).

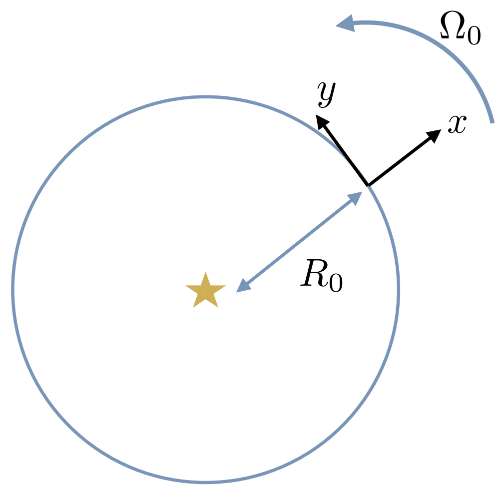

FIGURE 1 – Rotating frame on a circular orbit at R_0

Let us caracterise the dynamics of the flow in the vicinity of the fiducial point R_0, by assuming that $x, y, z \ll R_0$. For this, we have to know the effective gravity $\boldsymbol{g}$ in $\mathcal{R}'$. In the radial (x) direction, we have :

$$\begin{aligned}
g_x &= -\partial_R \psi + \Omega_0^2 R && (29)\\
&= -\frac{GMR}{(R^2+z^2)^{3/2}} + \Omega_0^2 R && (30)\\
&= -\frac{GM}{R^2} + O\Big(\frac{z^2}{R^2}\Big) + \Omega_0^2 R && (31)\\
&\simeq -\Omega_K^2(R)R + \Omega_0^2 R && (32)\\
&\simeq R\Big[\Omega_0^2 - \Omega_K(R_0+x)^2\Big] && (33)\\
&\simeq R\Bigg(\Omega_0^2 - \Big[\Omega_K(R_0) + x\frac{d\Omega_K}{dR} + O\Big(\frac{x^2}{R^2}\Big)\Big]^2\Bigg) && (34)\\
&\simeq -2R\Omega_K(R_0)x\frac{d\Omega_K}{dR} && (35)\\
&\simeq -2x\Omega_0^2\frac{d\log\Omega_K}{d\log R}. && (36)
\end{aligned}$$

It is customary to write $q = -d\log\Omega_K/d\log R$. In the case of a Keplerian disc, we have $q = 3/2$.

The equilibrium in the vertical direction is much simpler since no centrifugal acceleration is involved :

$$\begin{aligned}
g_z &= -\partial_z \psi && (37)\\
&= -\frac{GMz}{(R^2+z^2)^{3/2}} && (38)\\
&= -\frac{GMz}{R_0^3} + O(z^2/R^2) && (39)\\
&\simeq -\Omega^2 z. && (40)
\end{aligned}$$

The effective gravity in the rotating frame derives from a simple potential which may be written

$$\psi_{\mathrm{eff}} = \Omega^2\Big(-qx^2 + \frac{1}{2}z^2\Big) \qquad (41)$$

The effective potential therefore represents a saddle at the point $(x, z) = 0$, and it "looks" unstable in the radial (x) direction. Dynamically, as we will see, conservation of angular momentum (=Coriolis forces) prevents this from happening.

2.2 Local stability

2.2.1 Hydrodynamic stability

Let us now consider a particle evolving in the frame R' under the influence of the effective gravity and the Coriolis force. The particle is initially at rest at $(x, y) = 0$. Magnetic fields are neglected in this first approach.

The equation of motion for the fluid particle may be written

$$\frac{d^2x}{dt^2} = 2q\Omega_0^2 x + 2\Omega_0 \frac{dy}{dt} \tag{42}$$

$$\frac{d^2y}{dt^2} = -2\Omega_0 \frac{dx}{dt} \tag{43}$$

$$\frac{d^2z}{dt^2} = -\Omega_0^2 z \tag{44}$$

We first note that the vertical and horizontal equations of motion are separable. In the vertical direction, it describes oscillations of the fluid particle around the midplane at frequency Ω_0.

In the horizontal direction, the equations describes epicycles. To show it, let us first integrate (43) :

$$\mathcal{L} = \frac{dy}{dt} + 2\Omega_0 x \tag{45}$$

where $\mathcal{L}$ is a constant of motion. By looking back at the original (cylindrical) equations, we can see that $\mathcal{L}$ is nothing else but the local equivalent of the particle angular momentum. Our particle being initially in equilibrium at $(x, y) = 0$, it has $\mathcal{L} = 0$ and we can write the radial equation of motion as

$$\frac{d^2x}{dt^2} = (2q\Omega_0^2 - 4\Omega_0^2)x \tag{46}$$

hence, our effective gravitational potential which was initially unstable is stabilised thanks to the conservation of angular momentum, provided that $0 < q < 2$ ($q = 3/2$ for astrophysical discs).

The oscillations described by this particle have a frequency

$$\begin{aligned} \omega^2 &= 2\Omega_0^2(2 - q) \\ &\equiv \kappa^2 \end{aligned} \tag{47}$$

This characteristic frequency is named epicyclic frequency. In the particular case of a Keplerian disc ($q = 3/2$) we find $\kappa^2 = \Omega^2$ i.e. the epicyclic frequency coincide with the orbital frequency. As a result, orbits are closed, a well known property of the two body problem (e.g. Fig. 2)

2.2.2 Magnetohydrodynamic stability

As we have seen, protoplanetary discs are hydrodynamically stable at the linear level. In MHD, however, things start to be a bit more interesting.

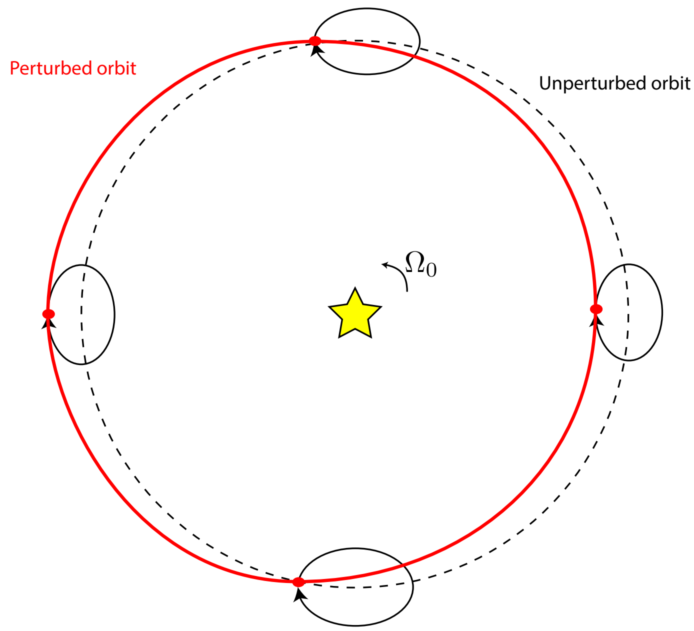

FIGURE 2 – Epicyclic oscillations of a fluid particle orbiting a point mass resulting in an closed elliptic orbit.

Let us plunge our disc in an external and constant magnetic field $\boldsymbol{B}_0$ which we assume is vertical. Assuming we still consider infinitesimal displacements around the equilibrium position of the fluid particles, the velocities are infinitely small, and the induction equation for magnetic fluctuations $\delta\boldsymbol{b}$ reads

$$\frac{\partial \delta\boldsymbol{b}}{\partial t} = B_0 \frac{\partial \boldsymbol{v}}{\partial z}. \tag{48}$$

Clearly, the stability will now depends on how we move the particles *with respect to each other*. Let us consider a set of particles initially at $(x, y) = 0$ and let us perturb these particle with a vertical harmonic perturbation

$$\boldsymbol{x} = \boldsymbol{x}_0 \exp(ikz) \tag{49}$$

the resulting magnetic perturbation will simply be

$$\frac{\partial \delta\boldsymbol{b}}{\partial t} = ikB_0\boldsymbol{v}, \tag{50}$$

which we can integrate in time

$$\delta\boldsymbol{b} = ikB_0\boldsymbol{x}. \tag{51}$$

In order to model how the field impacts the dynamics, we have to include the Lorentz force $\boldsymbol{F}_L$ in the equation of motion. In the horizontal direction, only the magnetic tension term $\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}$ appears, so we have

$$\frac{\boldsymbol{F}_L}{\rho} = \frac{\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\delta\boldsymbol{b}}{4\pi\rho} \tag{52}$$

$$= \frac{B_0\partial_z\delta\boldsymbol{b}}{4\pi\rho} \tag{53}$$

$$= -\frac{k^2B_0^2}{4\pi\rho}\boldsymbol{x} \tag{54}$$

$$= -V_A^2k^2\boldsymbol{x}, \tag{55}$$

where V_A is the Alfvén speed. The horizontal equation of motion are therefore reduced to

$$\frac{d^2x}{dt^2} = 2q\Omega_0^2 x + 2\Omega_0 \frac{dy}{dt} - V_A^2 k^2 x \tag{56}$$

$$\frac{d^2y}{dt^2} = -2\Omega_0 \frac{dx}{dt} - V_A^2 k^2 y \tag{57}$$

where it is clear that the magnetic forces are acting as a *restoring* force (hence the usual representation of a spring for the Lorentz force). Note also that angular momentum conservation is now broken by the azimuthal tension force. It is this effect which leads to an instability.

To show this, let us assume $\boldsymbol{x} = \boldsymbol{x} \exp(\sigma t)$. The equations of motion lead to the following eigenvalue problem

$$(\sigma^2 + V_A^2 k^2)x = 2q\Omega_0^2 x + 2\Omega_0 \sigma y \tag{58}$$

$$(\sigma^2 + V_A^2 k^2)y = -2\Omega_0 \sigma x \tag{59}$$

Which allows us to get the dispersion relation :

$$(\sigma^2 + V_A^2 k^2)^2 - 2q\Omega_0^2(\sigma^2 + V_A^2 k^2) + 4\Omega_0^2 \sigma^2 = 0 \tag{60}$$

which allows us to recover epicyclic oscillations when $V_A = 0$ with $\sigma^2 = -2\Omega_0^2(2-q) = -\kappa^2$ and pure Alfvénic oscillations when $\Omega_0 = 0$ with $\sigma^2 = -V_A^2 k^2$.

Expanding this dispersion relation leads to

$$\sigma^4 + \sigma^2\Big(\kappa^2 + 2V_A^2 k^2\Big) + V_A^2 k^2 \Big(V_A^2 k^2 - 2q\Omega_0^2\Big) = 0 \tag{61}$$

This dispersion relation leads to an instability when σ^2 is real, i.e. when

$$V_A^2 k^2 - 2q\Omega_0^2 < 0. \tag{62}$$

This instability is the magneto-rotational instability (or MRI). It works when the magnetic tension force is not too strong, as suggested by (62). It is possible to solve the full dispersion analytically to get the eigenvalues (see Fig. 3). When $V_A k < \sqrt{2q}\Omega_0$, positive eigenvalues are found which are the signature of the MRI. The maximum growth rates are obtained for $V_A k = \sqrt{2q}\Omega_0/2$ with $\sigma_{\max} = q\Omega/2$. Above the limit (**??**), the unstable branch becomes an stable Alfvén wave, which shows that the MRI is mostly an Alfvénic perturbation. In addition to this pair of branches, we find a pair of epicyclic modes which are stable for all kV_A (Fig. **??**), and have a non-zero frequency for $V_A = 0$.

The physical interpretation of the MRI is straightforward : consider 2 fluid particles attached to a vertical field line and assume we slightly move these particles radially. At first, they will start an epicyclic motion and drift azimuthally (Fig.4). As they drift away, however, the azimuthal magnetic tension will act as a spring bringing back the particles together, slowing down the inner particle and accelerating the outer particle. This results in a loss of angular momentum for the inner particle, which falls further down, and reversely for the outer particle. This mechanism can only work if the radial magnetic tension is sufficiently weak (as stated by eq. 62). Otherwise, the particles come back to their initial point resulting in an Alfvénic oscillation.

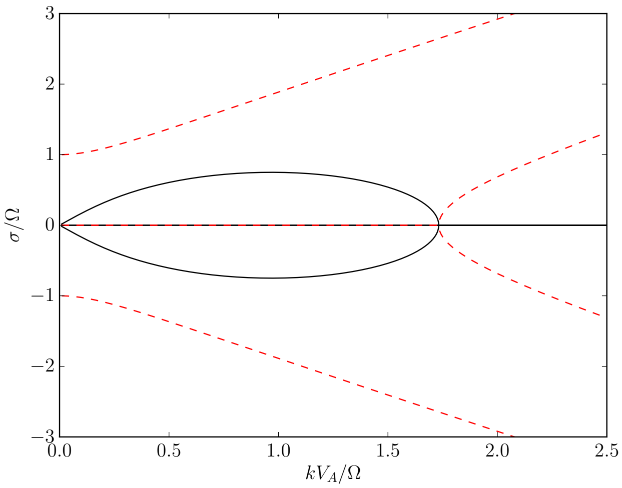

FIGURE 3 – Real part (black) and imaginary part (red dashed line) of the solutions of (61) with $q = 3/2$. The MRI appears for weak enough fields $V_A k < \sqrt{3}\Omega_0$.

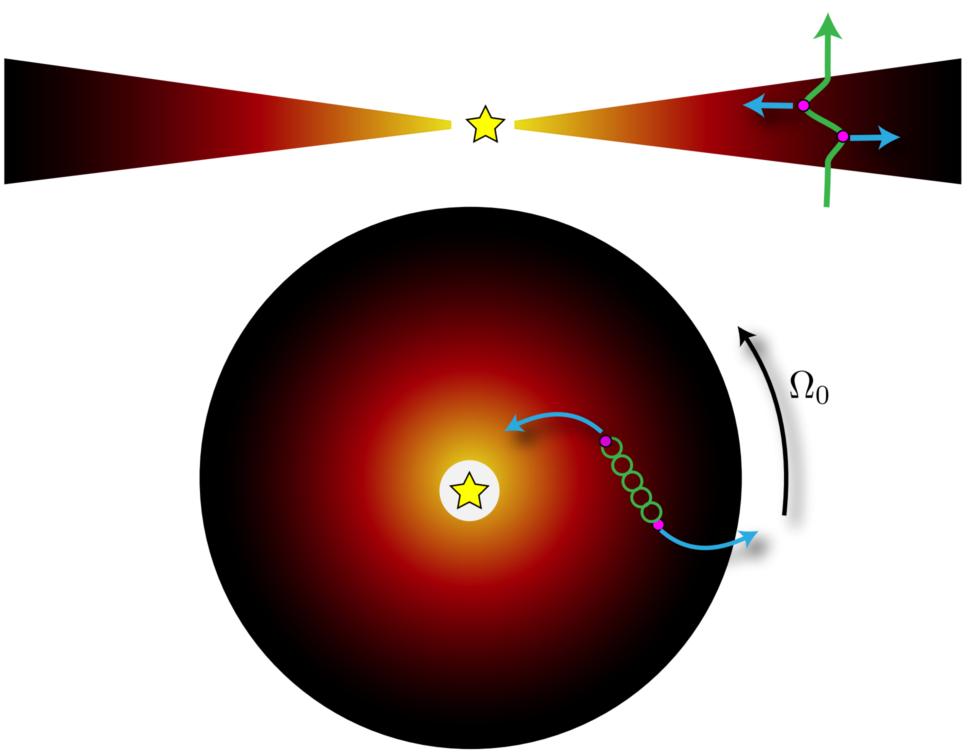

FIGURE 4 – Physical representation of the MRI mechanism (see text).