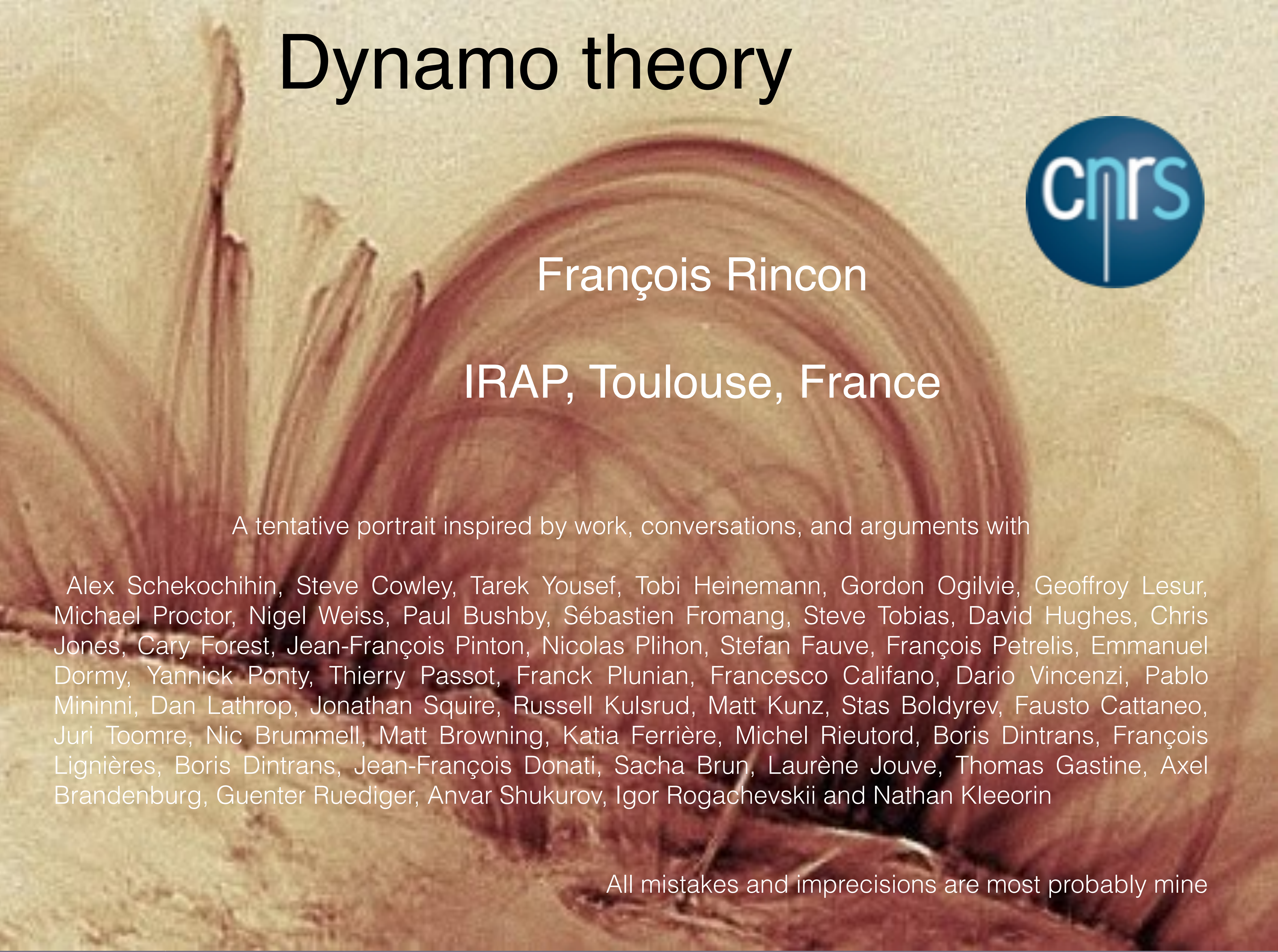


Dynamo theory

François Rincon

IRAP, Toulouse, France

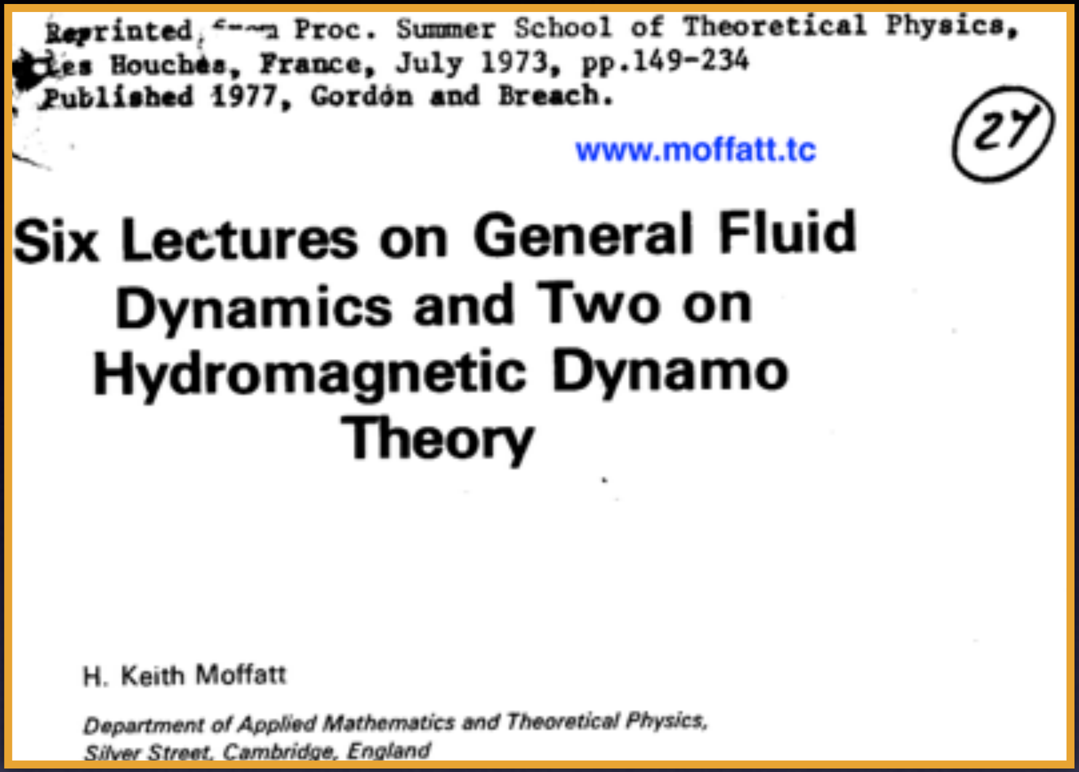
A tentative portrait inspired by work, conversations, and arguments with

Alex Schekochihin, Steve Cowley, Tarek Yousef, Tobi Heinemann, Gordon Ogilvie, Geoffroy Lesur, Michael Proctor, Nigel Weiss, Paul Bushby, Sébastien Fromang, Steve Tobias, David Hughes, Chris Jones, Cary Forest, Jean-François Pinton, Nicolas Plihon, Stefan Fauve, François Petrelis, Emmanuel Dormy, Yannick Ponty, Thierry Passot, Franck Plunian, Francesco Califano, Dario Vincenzi, Pablo Mininni, Dan Lathrop, Jonathan Squire, Russell Kulsrud, Matt Kunz, Stas Boldyrev, Fausto Cattaneo, Juri Toomre, Nic Brummell, Matt Browning, Katia Ferrière, Michel Rieutord, Boris Dintrans, François Lignières, Boris Dintrans, Jean-François Donati, Sacha Brun, Laurène Jouve, Thomas Gastine, Axel Brandenburg, Guenter Ruediger, Anvar Shukurov, Igor Rogachevskii and Nathan Kleeorin

All mistakes and imprecisions are most probably mine

Talk outline

- Introduction
  - Short and easy (3h)
- Setting the stage
  - Not too long and “straightforward” (4h)
- Small scale dynamos
  - Long and difficult* (6h)
- Large-scale dynamos
  - Just a tad shorter and less difficult* (4h)
- Connections between the two
  - Short and controversial (2h)
- Instability-driven dynamos
  - Short and seemingly easier, but actually difficult* (2h)
- Collisionless plasma dynamo
  - Short and a bit crazy** (1h)



Reprinted from Proc. Summer School of Theoretical Physics, Les Houches, France, July 1973, pp.149-234
Published 1977, Gordon and Breach.

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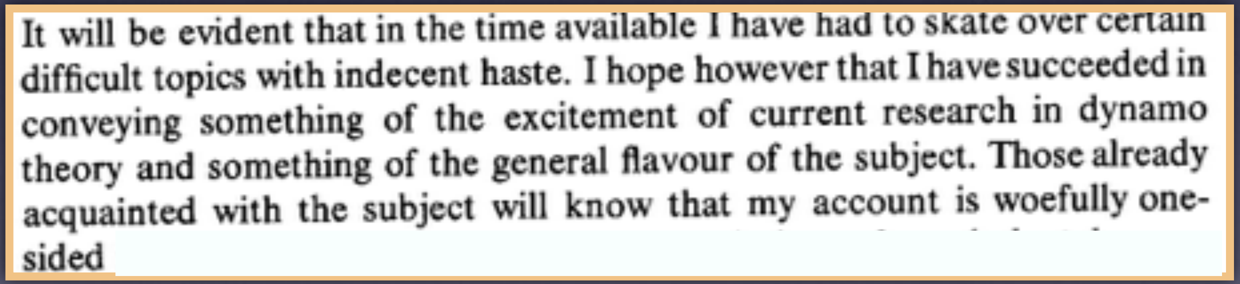
www.moffatt.tc

Six Lectures on General Fluid Dynamics and Two on Hydromagnetic Dynamo Theory

H. Keith Moffatt

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge, England



It will be evident that in the time available I have had to skate over certain difficult topics with indecent haste. I hope however that I have succeeded in conveying something of the excitement of current research in dynamo theory and something of the general flavour of the subject. Those already acquainted with the subject will know that my account is woefully one-sided

*including for the lecturer

**including the lecturer

Introduction

What is dynamo theory about ?

- The origin, and sustainment, of magnetic fields in the universe
  - on the Earth, other planets and their satellites ("planetary magnetism")
  - on the Sun and other stars ("stellar magnetism")
  - in galaxies, clusters and the early universe ("cosmic magnetism")
- Understanding their structural, statistical, and dynamical properties
- Addressing important physics (and maths) problems
  - Deep connections with hydrodynamic turbulence and more generally turbulent transport problems
- Coming up with "useful stuff" for experimentalists and observers
  - Warning: people have strong disagreements on the definition of "useful stuff"

The fluid/plasma dynamo conundrum

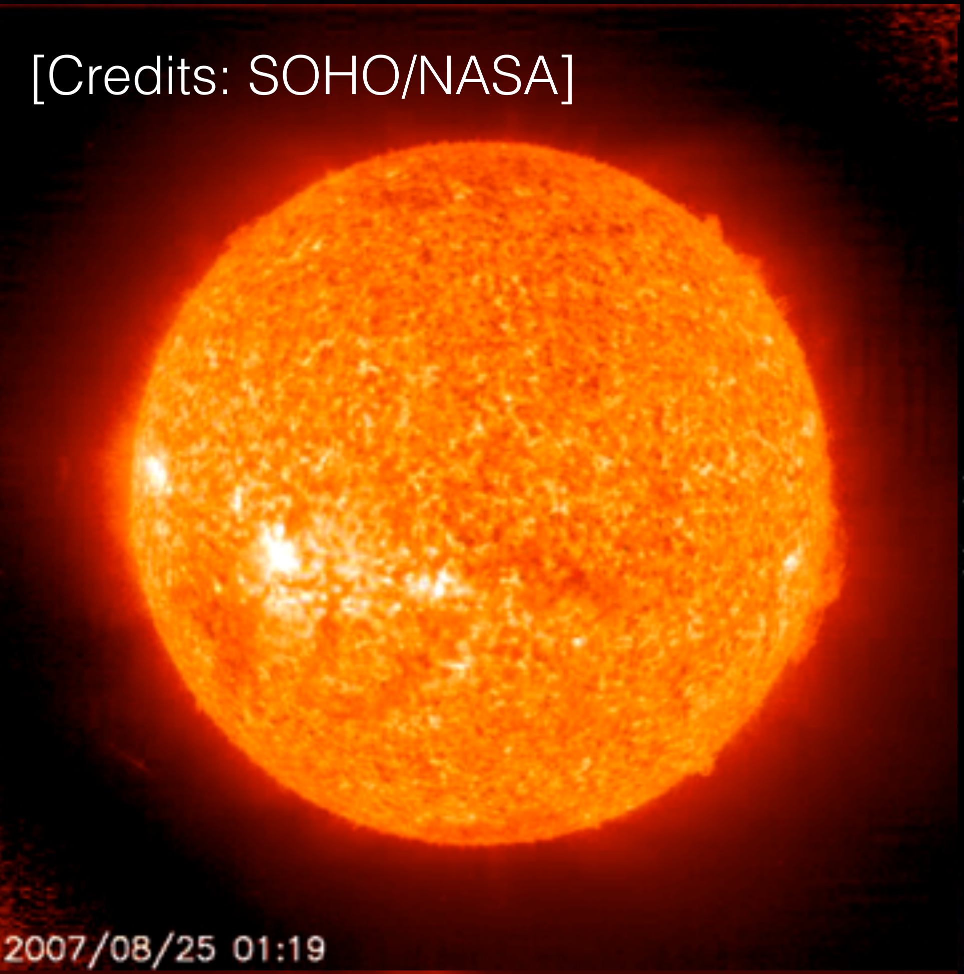
- Most astrophysical bodies, and many planetary interiors, are
  - in an electrically conducting fluid (MHD) or weakly-collisional plasma state
  - in a turbulent state
  - (differentially) rotating: shearing, Coriolis and precessing effects
- Main questions
  - Can flows of electrically conducting fluid/plasma amplify magnetic fields ?
  - What are at the time and spatial scales on which this happens ?
  - At what amplitude do they saturate ? What field structure is produced ?
- A complex and multifaceted problem
  - Requires observations, phenomenology, theory, numerics and experiments

A touch of history

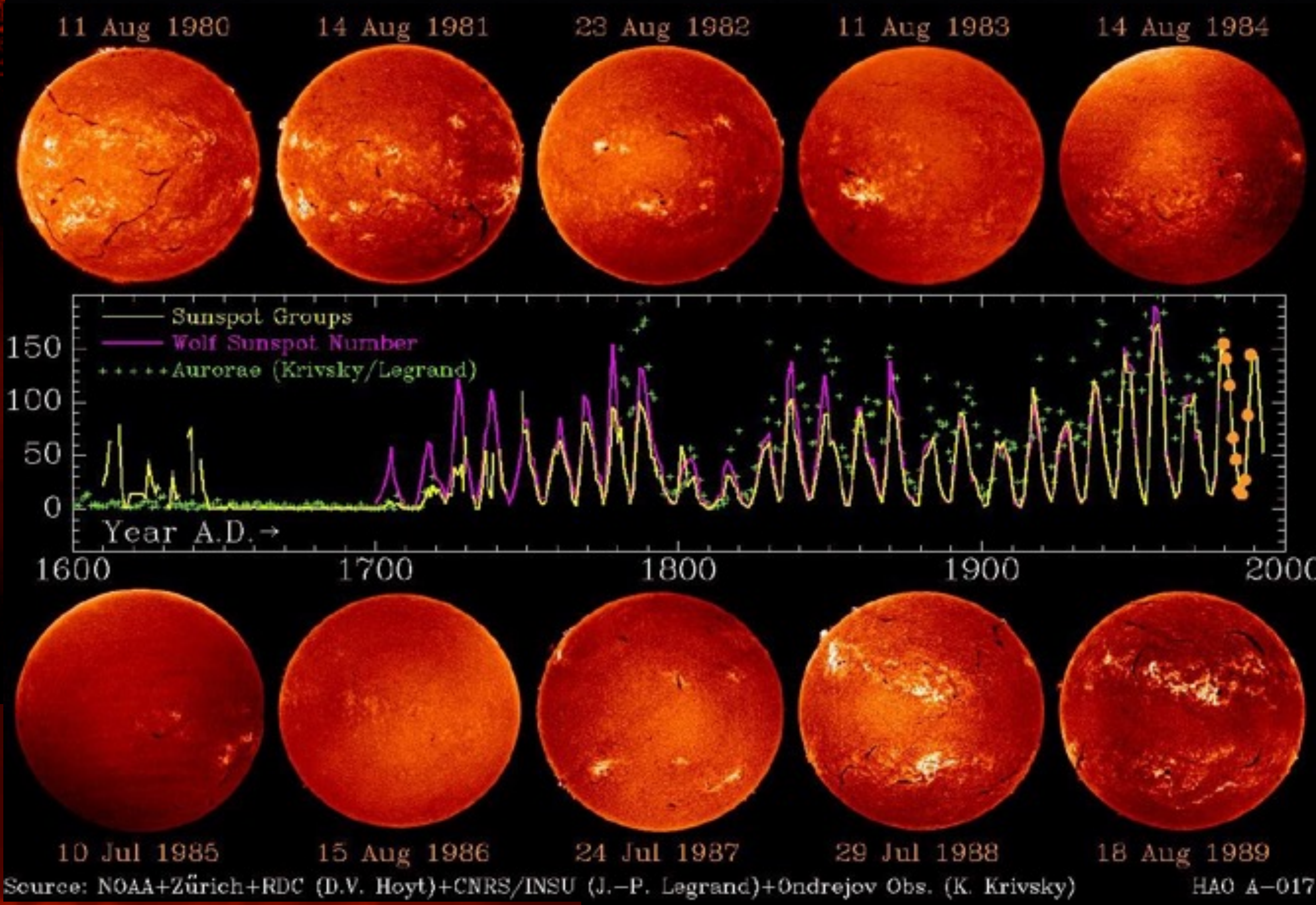
- Self-exciting fluid dynamos will soon be a century-old idea
  - First invoked by Larmor in 1919 (sunspot magnetism)

- The idea took a lot of time to gain ground
  - Cowling's antidynamo theorem (1933)
  - First examples in the 1950s (e.g. Herzenberg dynamo)
  - Parker's solar dynamo phenomenology (1955)
- Mathematical theory
  - Alpha effect / mean-field: Steenbeck, Krause, Raedler 1966, Moffatt, Roberts etc. (1970s)
  - Small-scale dynamo theory: Kazantsev 1967, Kraichnan, Zel'dovich et al. (70s-80s)
- Numerical and experimental era
  - Numerical evidence of turbulent dynamos: Meneguzzi et al. 1981, flourishing since then
  - Experimental evidence: Riga, Karlsruhe (~2000), VKS (2007), plasma underway (2005+)
  - Great observational radio and spectro-polarimetric prospects too (stellar, galactic, cosmo)

Solar magnetism

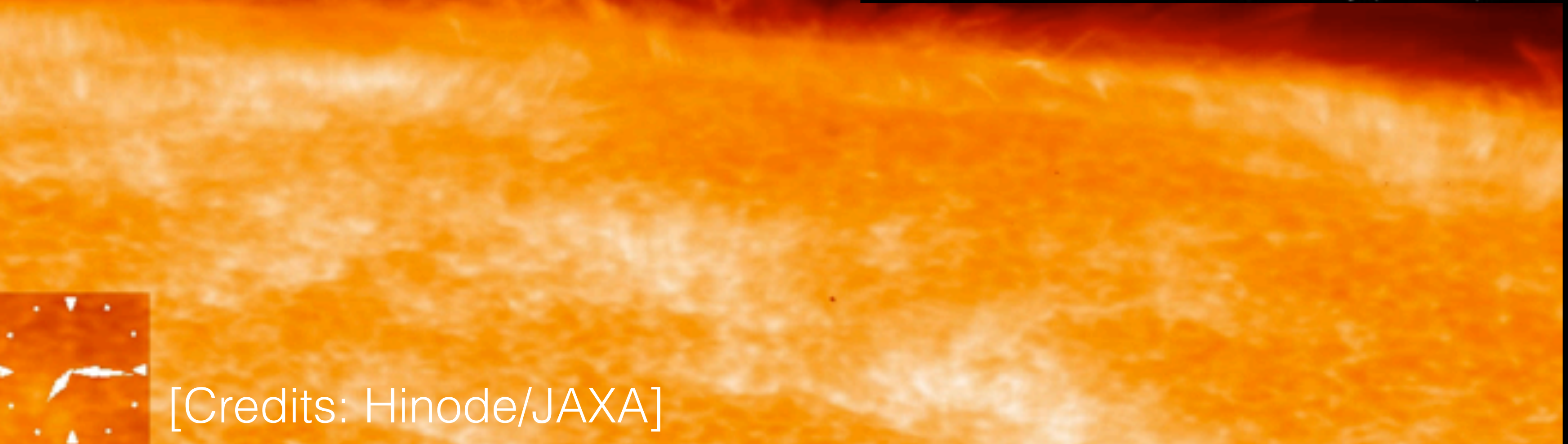


[Credits: SOHO/NASA]



Source: NOAA+Zürich+RDC (D.V. Hoyt)+CNRS/INSU (J.-P. Legrand)+Ondrejov Obs. (K. Krivsky)

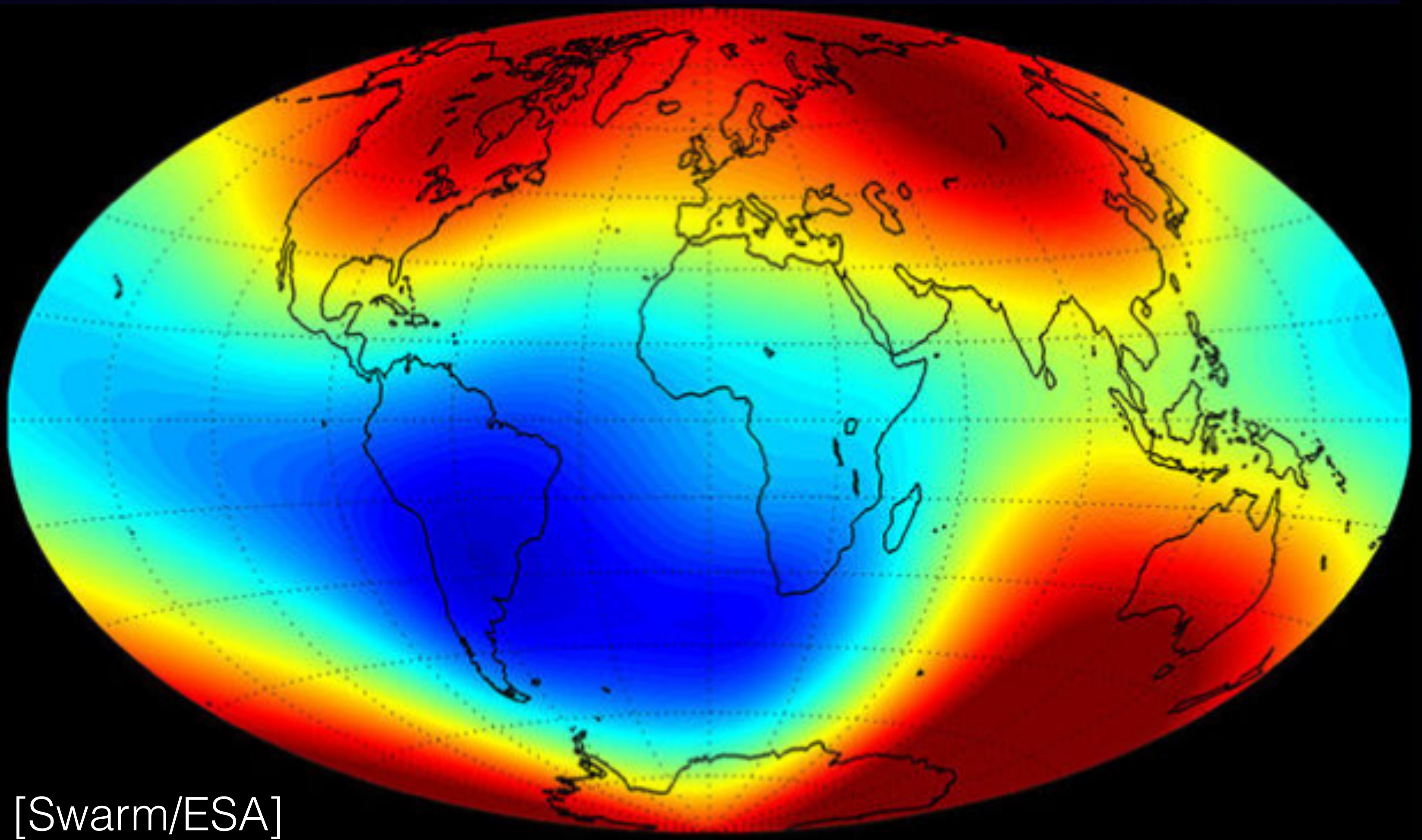
Global solar cycle dynamics



[Credits: Hinode/JAXA]

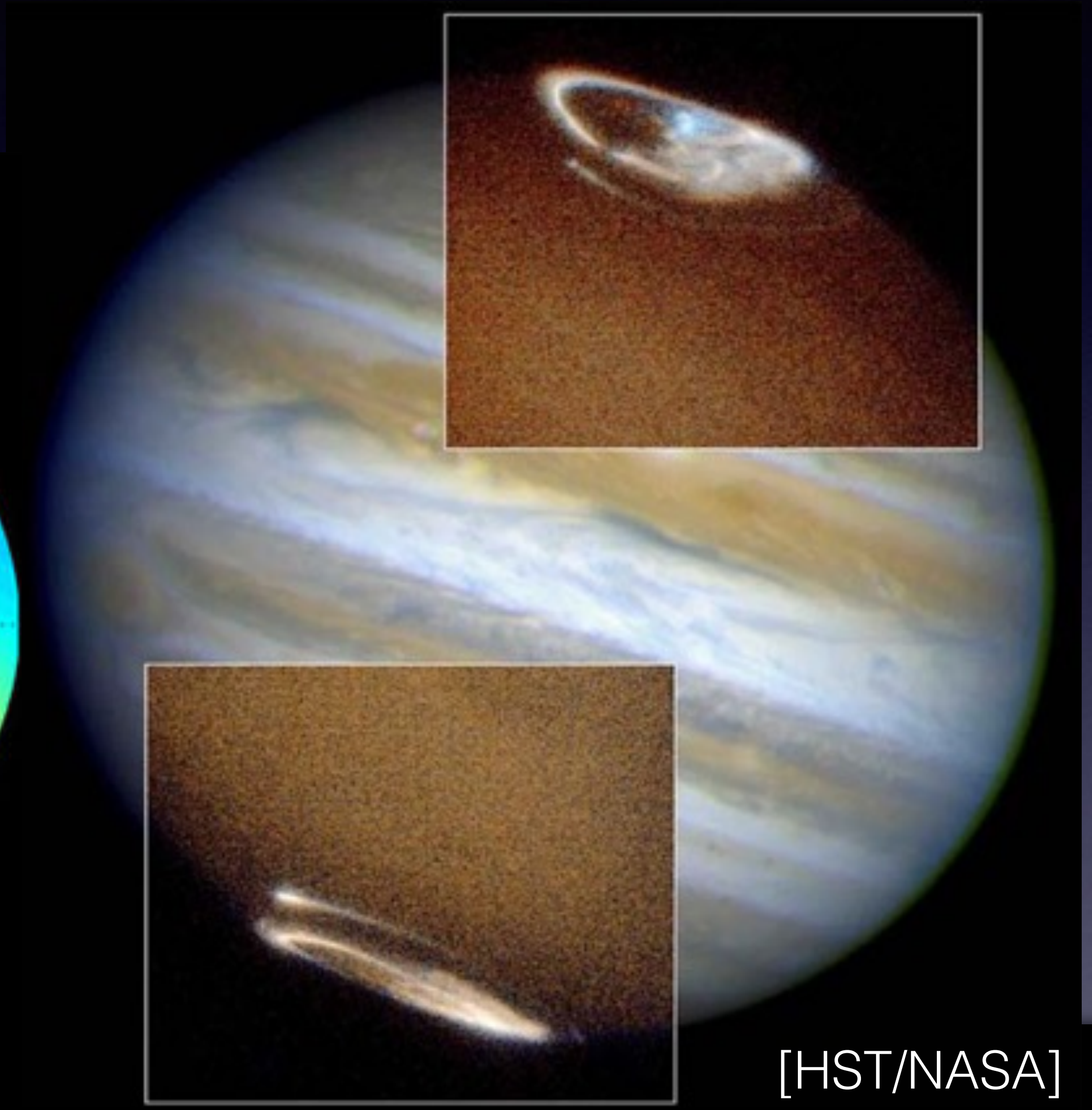
Small-scale surface dynamics

Planetary magnetism



[Swarm/ESA]

Earth's magnetic field (2014)

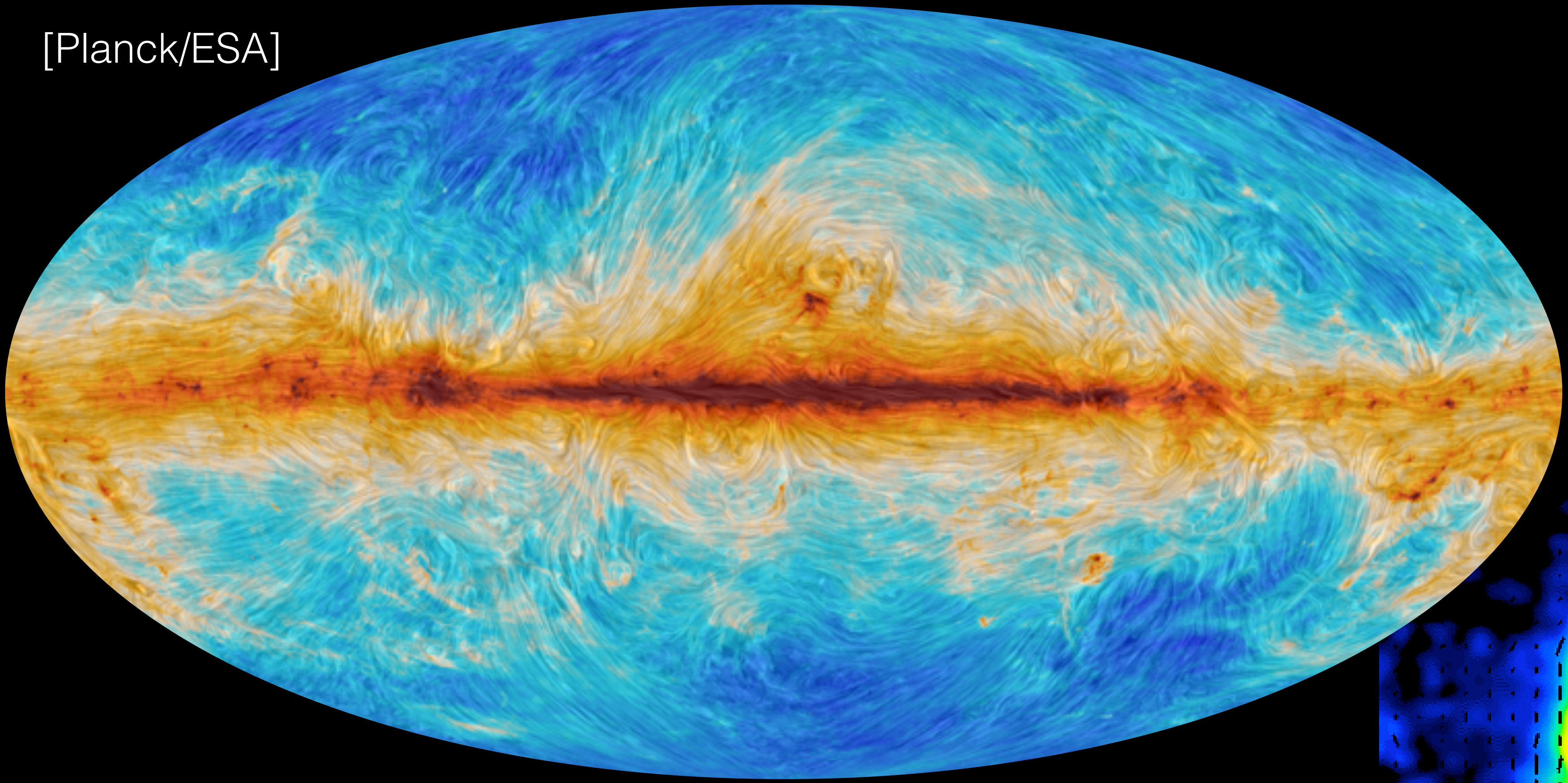


[HST/NASA]

Jupiter Auroras

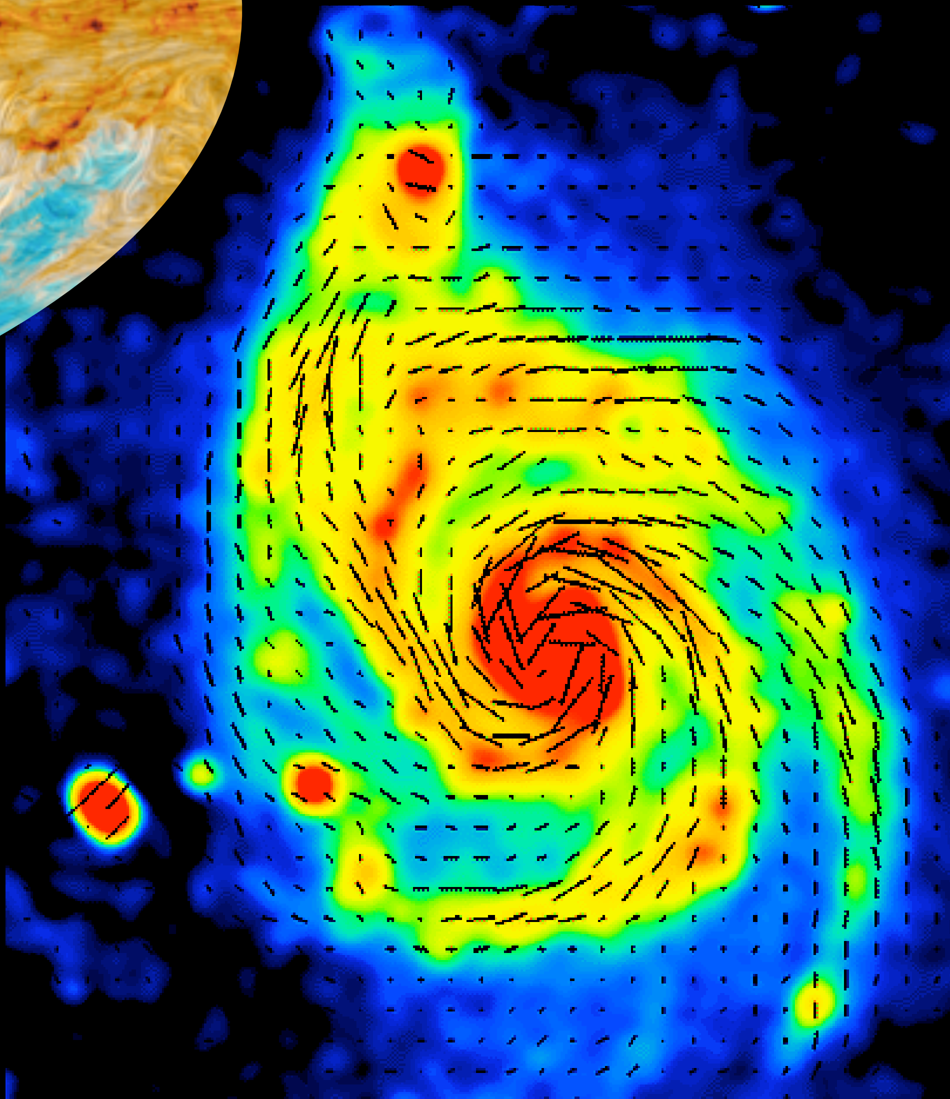
Galactic magnetism

[Planck/ESA]



Galactic magnetic field

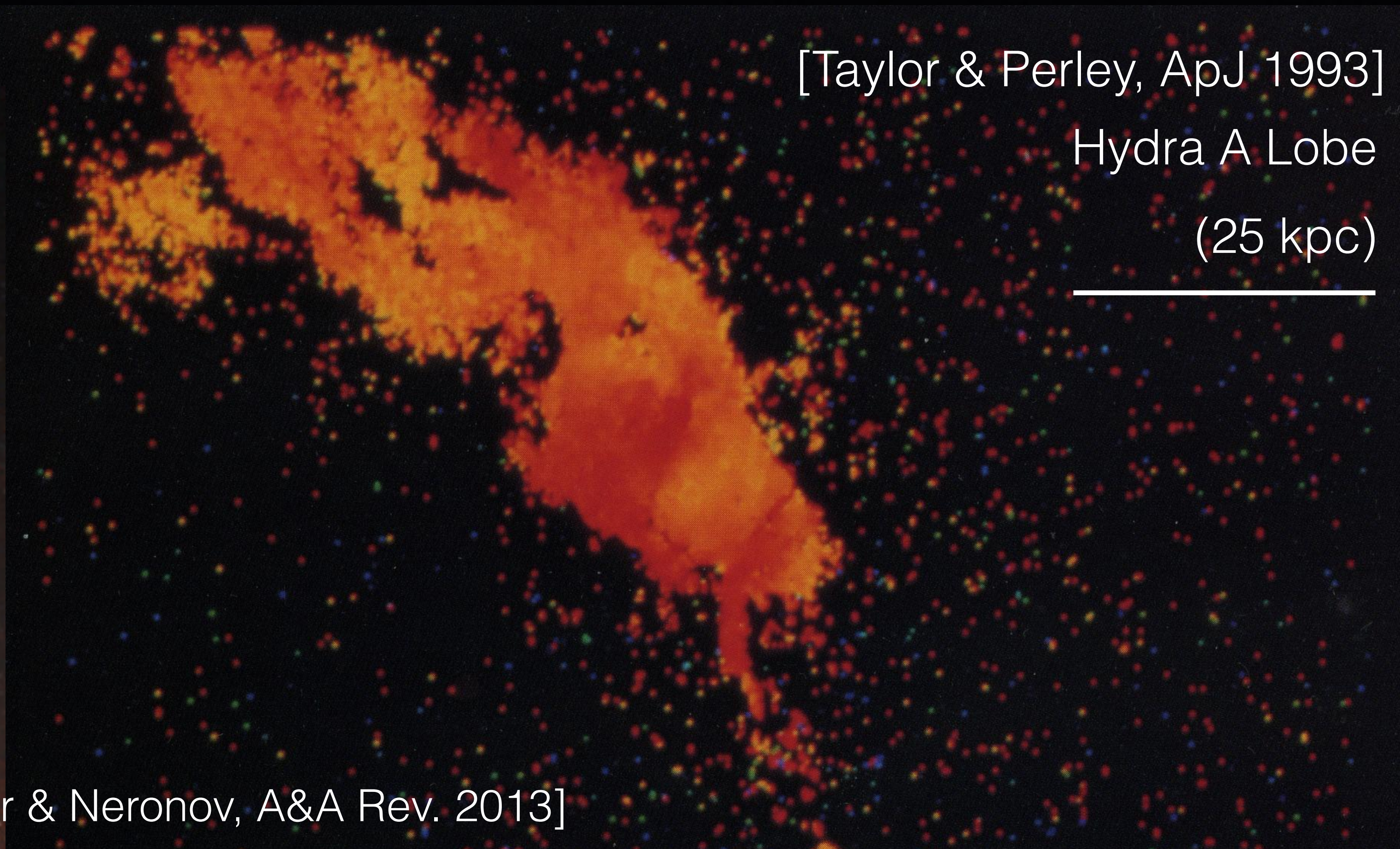
M51 magnetic field



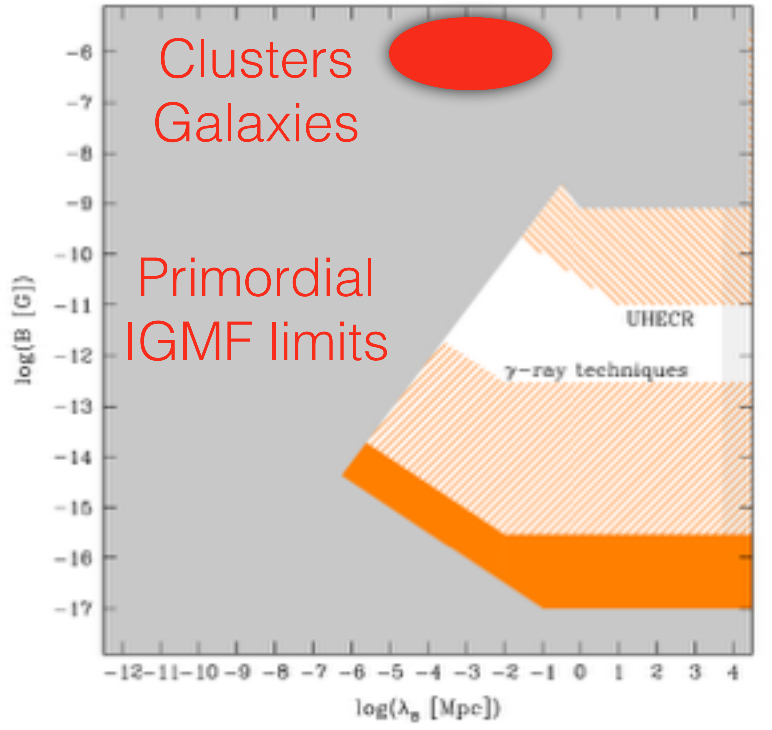
[Beck et al. VLA/Effelsberg]

Galaxy clusters and cosmic magnetism


[Fabian et al. ESA/NASA]
Perseus/NGC 1275 filaments


[Taylor & Perley, ApJ 1993]
Hydra A Lobe
(25 kpc)

[Durrer & Neronov, A&A Rev. 2013]



Takeaway phenomenological points

- Many astrophysical objects have global, ordered fields
  - Differential rotation, global symmetries and geometry important
  - Coherent structures and MHD instabilities may also be very important
  - Motivation for the development of “large-scale” dynamo theories

- Lots of “small-scale”, random fields also discovered from the 70s
  - These come hand in hand with global magnetism
  - Simultaneous development of “small-scale dynamo” theory

- Astrophysical magnetism is in a nonlinear, saturated state
  - Linear theory likely not the whole story (or requires non-trivial justification)
  - Multiple scale interactions expected to be important

Setting the stage

Mathematical formulation

- Compressible, viscous, resistive MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Lorentz force

External forcing (spoon, gravity etc.)

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{\mathbf{j} \times \mathbf{B}}{c} + \nabla \cdot \boldsymbol{\tau} + \mathbf{F}(\mathbf{x}, t)$$

Viscous stresses

$$\boldsymbol{\tau}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right)$$

Electromotive force

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B})$$

Magnetic diffusion $\eta = \frac{c^2}{4\pi\sigma}$

$$\nabla \cdot \mathbf{B} = 0 \qquad \mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B}$$

Dissipation

Thermal diffusion

$$\rho T \left(\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s \right) = D_\mu + D_\eta + \nabla \cdot (\kappa \nabla T)$$

Magnetic field energetics

- Magnetic energy equation

$$\frac{d}{dt}\int \frac{\mathbf{B}^2}{8\pi}dV = -\int \mathbf{u}\cdot\frac{(\mathbf{j}\times\mathbf{B})}{c}dV - \frac{c}{4\pi}\oint(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S} - \int\frac{\mathbf{j}^2}{\sigma}dV$$

Minus the work of the Lorentz force on the flow

Poynting flux

Ohmic dissipation

  - Magnetic field is generated at the expense of kinetic energy

- Simple but enlightening local equation (ideal MHD)

$$\frac{1}{B}\frac{DB}{Dt} = \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{u} - \nabla\cdot\mathbf{u}$$

Stretching rate

Compression rate

$$\hat{\mathbf{b}} = \frac{\mathbf{B}}{B}$$

Conservation laws in ideal MHD

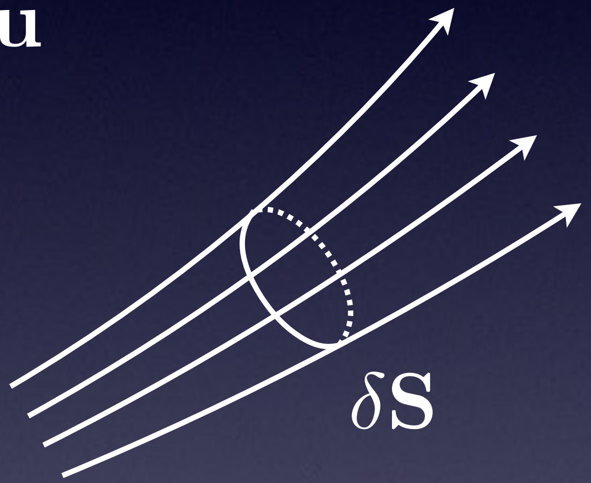
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$$

- Alfvén's theorem(s)
  - Magnetic field lines are "frozen into" the fluid just as material lines

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right) = \frac{\mathbf{B}}{\rho}\cdot\nabla\mathbf{u} \qquad \frac{D\delta\mathbf{r}}{Dt} = \delta\mathbf{r}\cdot\nabla\mathbf{u}$$

  - Magnetic flux through material surfaces is conserved

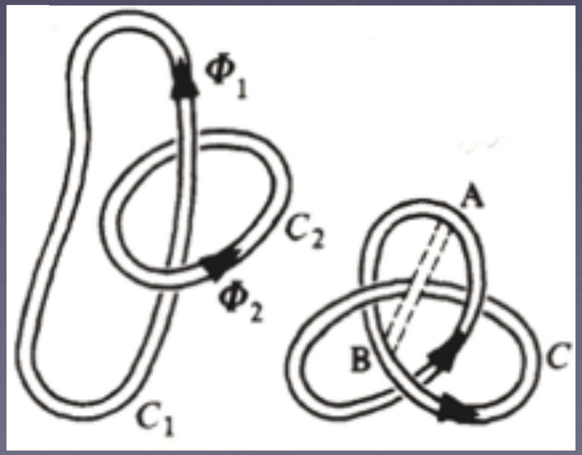
$$\frac{D}{Dt}(\mathbf{B}\cdot\delta\mathbf{S}) = 0$$



- Magnetic helicity $\mathcal{H}_m = \int \mathbf{A}\cdot\mathbf{B}\, d^3\mathbf{r}$ conservation
  - A measure of magnetic linkage / knottedness

$$\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\mathbf{E} - \nabla\varphi$$

$$\frac{\partial}{\partial t}(\mathbf{A}\cdot\mathbf{B}) + \nabla\cdot[c\varphi\mathbf{B} + \mathbf{A}\times(\mathbf{u}\times\mathbf{B})] = 0$$



Simplest MHD system for dynamo theory

- Incompressible, resistive, viscous MHD
  - Captures a great deal of the dynamo problem

Magnetic tension

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \mathbf{B}\cdot\nabla\mathbf{B} + \nu\Delta\mathbf{u} + \mathbf{f}(\mathbf{x},t)$$

Induction

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}$$

$$P = p + \frac{B^2}{2}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \nabla\cdot\mathbf{B} = 0$$

p and $\mathbf{B}$ rescaled by ρ and $(4\pi\rho)^{1/2}$

- Often paired with simple periodic boundary conditions
  - Problematic in some cases (more later)

Scales and dimensionless numbers

- System/integral scale ℓ_0, U_0
- Fluid system with two dissipation channels
  - Dimensionless numbers:

$$\mathrm{Re} = \frac{\ell_0 U_0}{\nu} \qquad \mathrm{Rm} = \frac{\ell_0 U_0}{\eta} \qquad \mathrm{Pm} = \frac{\nu}{\eta}$$

  - Kolmogorov viscous scale $\ell_\nu \sim \mathrm{Re}^{-3/4}\, \ell_0$, $u_\nu \sim \mathrm{Re}^{-1/4}\, U_0$
  - Magnetic resistive scale ℓ_η (Pm-dependent)
- Another important dimensionless quantity
  - Eddy turnover time $\tau_{\mathrm{NL}} \sim \ell_u/u$
  - Flow/eddy correlation time τ_c

$$\mathrm{St} = \frac{\tau_c}{\tau_{\mathrm{NL}}}$$

Strouhal/Kubo number

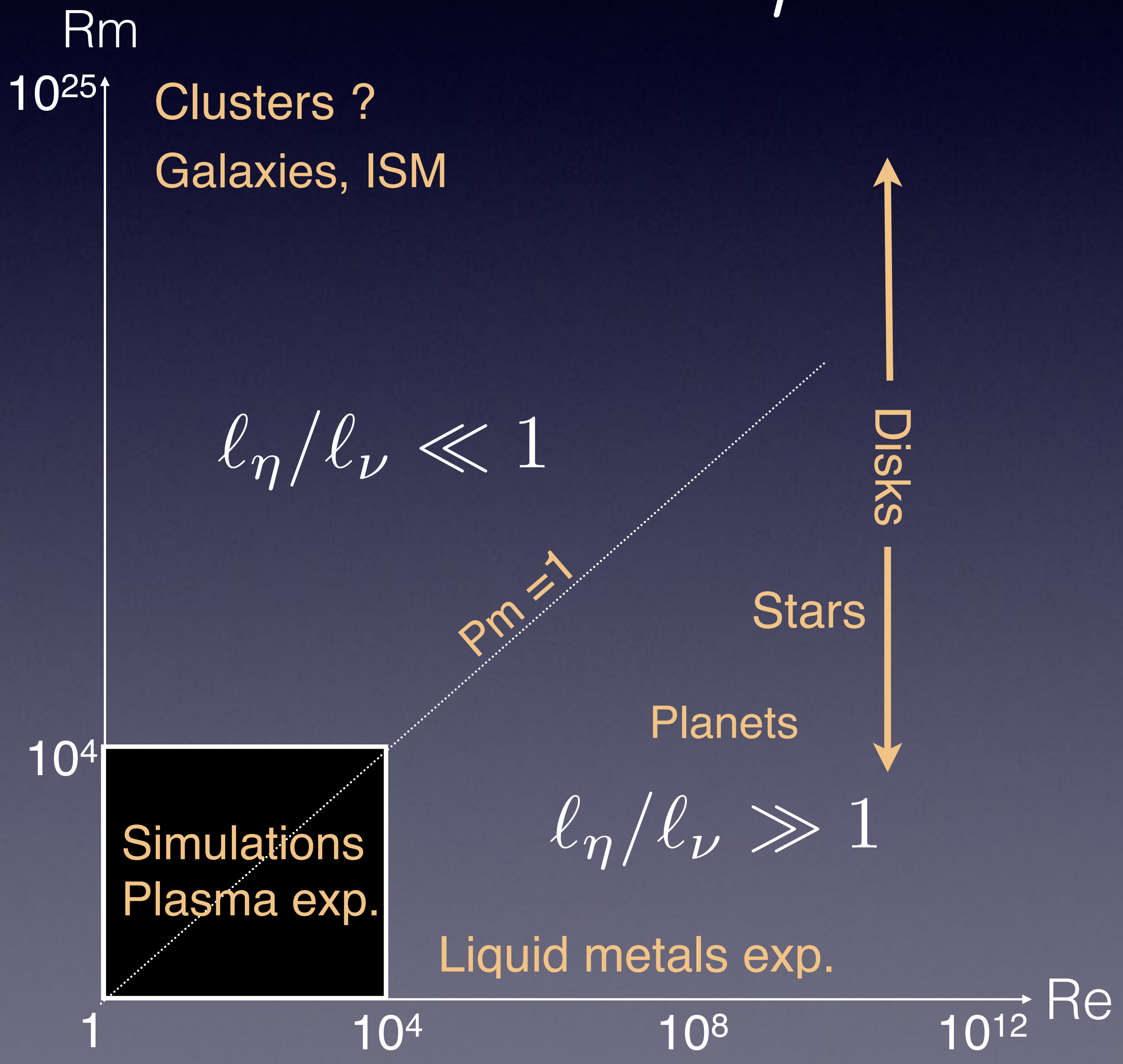
The magnetic Prandtl number landscape

- Wide range of Pm in nature
  - Liquid metals have Pm << 1
  - Computers have Pm ~ O(1)
- For a collisional hydrogen plasma [Te=Ti in K, *n* in S.I.]

$$\mathrm{Pm} = 2.5 \times 10^3 \frac{T^4}{n \ln \Lambda^2}$$

- Pm<1 and Pm>1 seemingly very different situations
  - Naively, Pm>1 makes life easier to magnetic fields

$$\mathrm{Pm} = \frac{\nu}{\eta}$$

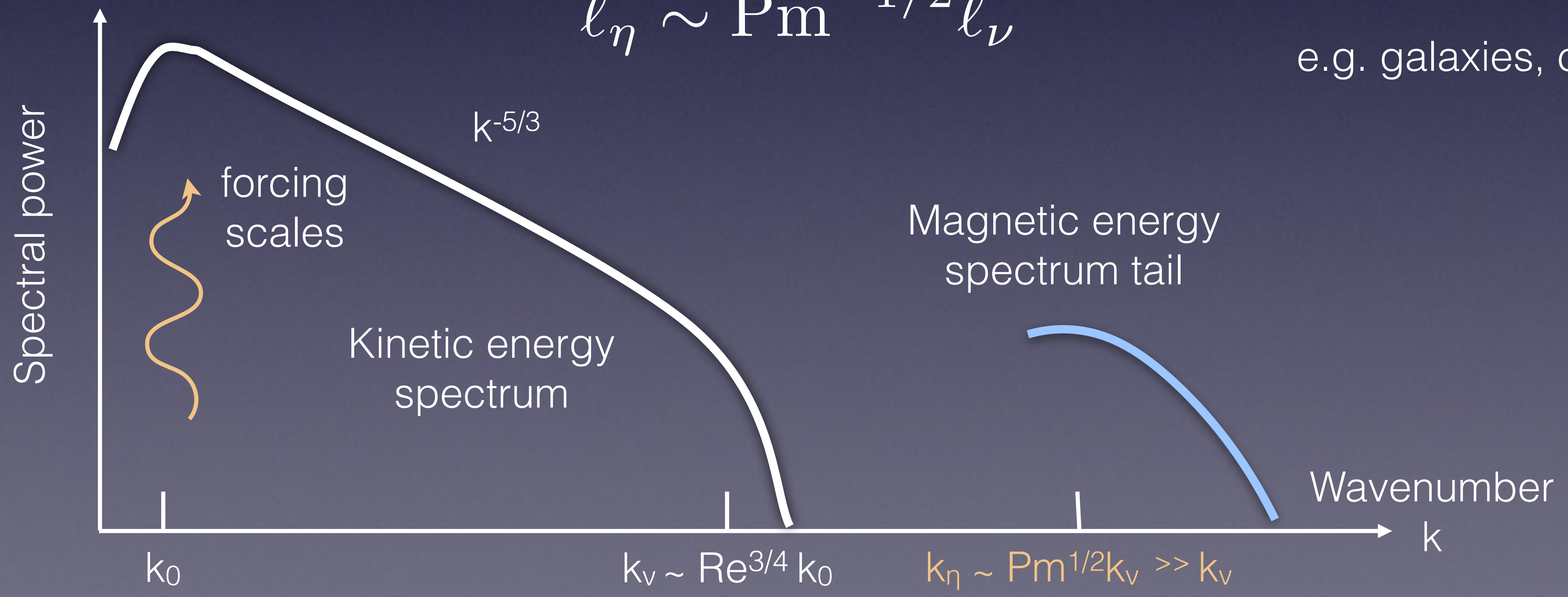


Large magnetic Prandtl numbers

- Pm > 1: resistive cut-off scale is smaller than viscous scale
  - In Kolmogorov turbulence, rate of strain goes as $\ell^{-2/3}$
  - Viscous eddies are the fastest at stretching B: $u_\nu / \ell_\nu \sim \mathrm{Re}^{1/2}\, U_0 / \ell_0$
  - To estimate the resistive scale ℓ_η, balance stretching by these eddies ~ u_ν/ℓ_ν with ohmic diffusion rate η/ℓ_η^2

$$\ell_\eta \sim \mathrm{Pm}^{-1/2} \ell_\nu$$

e.g. galaxies, clusters

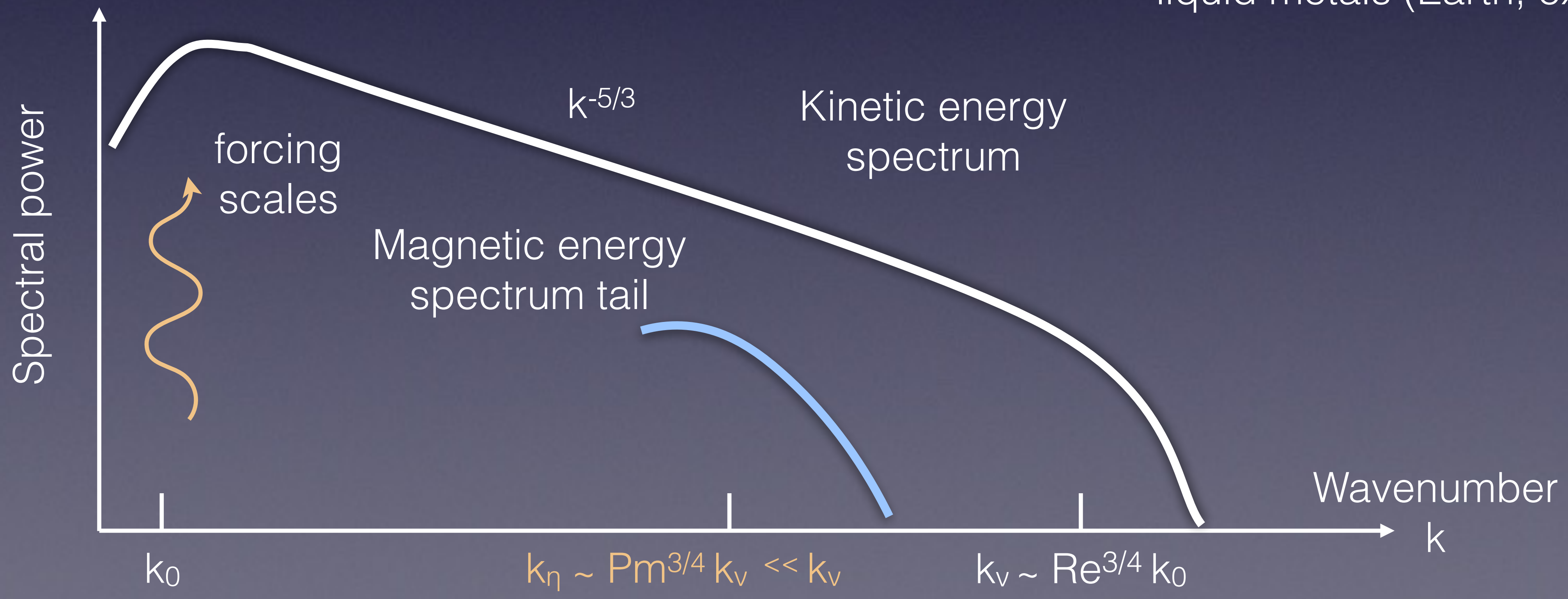


Low magnetic Prandtl numbers

- Pm < 1: resistive cut-off falls in the turbulent inertial range
  - To estimate the resistive scale ℓ_η, balance magnetic stretching by the eddies at the same scale ~ u_η/ℓ_η, with diffusion η/ℓ_η^2
  - i.e., $\mathrm{Rm}(\ell_\eta) = u(\ell_\eta)\,\ell_\eta / \eta \sim 1$

$$\ell_\eta \sim \mathrm{Pm}^{-3/4}\ell_\nu$$

e.g. stellar, solar,
liquid metals (Earth, experiments)

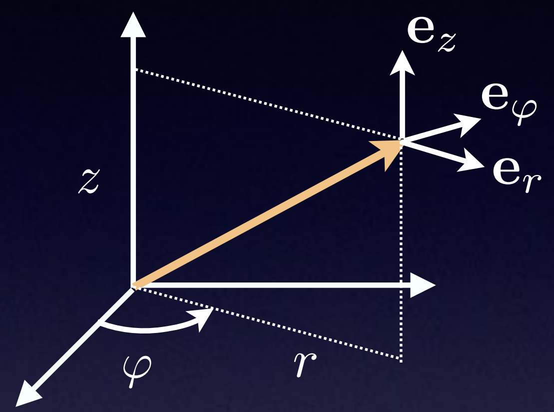


Dynamo fundamentals

- The problem of exciting a dynamo is an instability problem
    - Growth requires stretching to overcome diffusion (measured by $\mathrm{Rm} = \dfrac{\ell_0 U_0}{\eta}$)
- Kinematic dynamo problem: $\dfrac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}$
    - Find exponentially growing solutions of the linear induction equation (velocity field is prescribed)
- Dynamical problem considers effects of Lorentz force on $\mathbf{u}$
    - Saturated state of kinematic dynamos: non-linear magnetic back reaction
    - Subcritical scenarios: e.g. joint excitation of u and B via MHD instabilities
- Slow vs Fast
    - A dynamo is slow/fast if its growth rate does/doesn't vanish as $\eta \to 0$

Cowling's antidynamo theorem

- Axisymmetric dynamo action is impossible [Cowling, MNRAS, 1933]
  - In polar geometry, write



Poloidal Toroidal

  - $\mathbf{B} = \nabla \times (\chi \mathbf{e}_\varphi / r) + r\psi\, \mathbf{e}_\varphi$
  - $\mathbf{u} = \mathbf{u}_{\mathrm{pol}} + r\Omega\, \mathbf{e}_\varphi$

$$\frac{\partial \chi}{\partial t} + \mathbf{u}_{\mathrm{pol}} \cdot \nabla \chi = \eta \left(\Delta - \frac{2}{r}\frac{\partial}{\partial r} \right) \chi$$

No source term

$$\frac{\partial \psi}{\partial t} + \mathbf{u}_{\mathrm{pol}} \cdot \nabla \psi = \mathbf{B}_{\mathrm{pol}} \cdot \nabla \Omega + \eta \left(\Delta + \frac{2}{r}\frac{\partial}{\partial r} \right) \psi$$

  - Poloidal flow can only redistribute flux so χ must decay ultimately
  - As χ decays, so must the toroidal field
- Note: only applies if u and B share the same symmetry axis

Antidynamo theorems and their implications

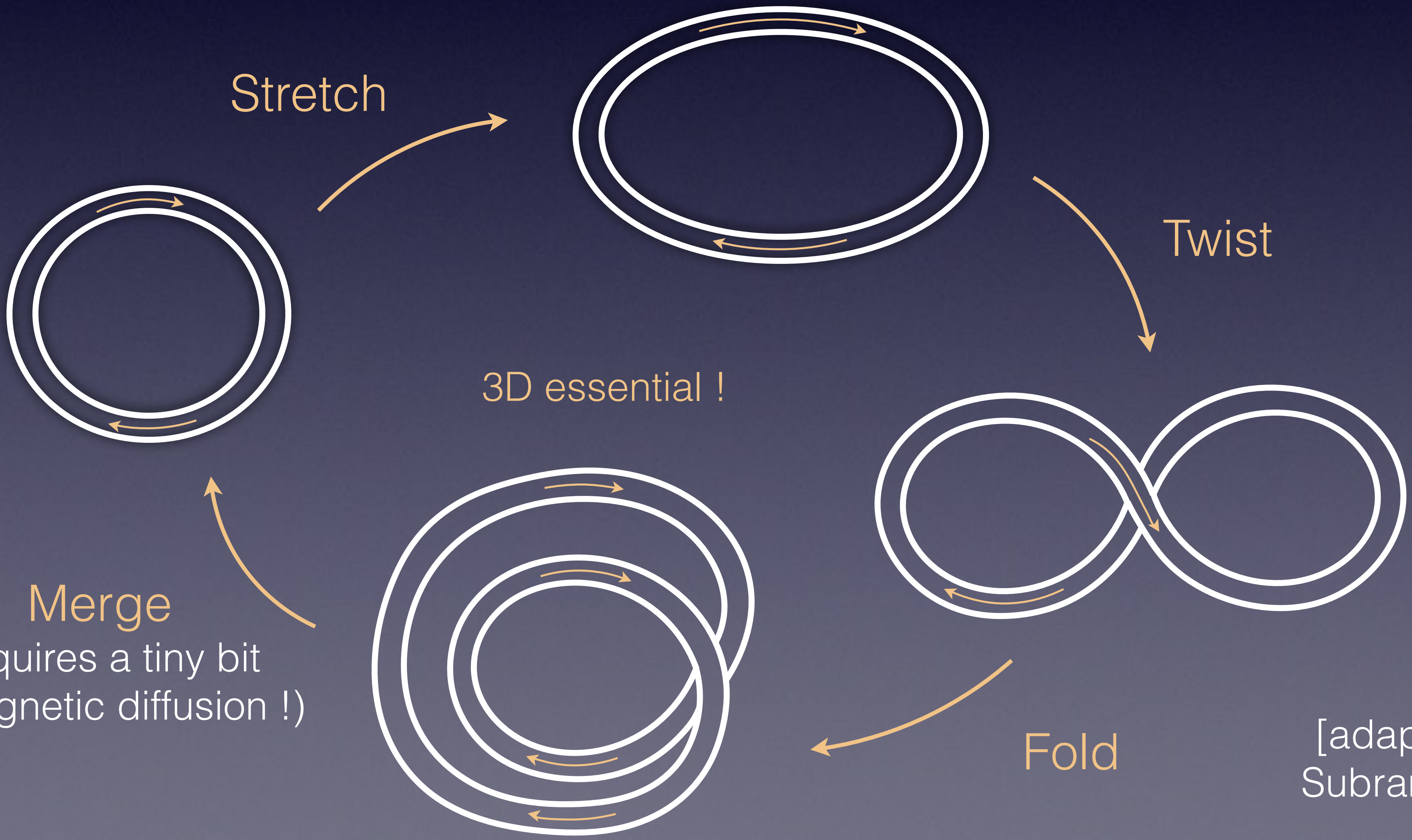
- Many other antidynamo results can be proven
  - Plane two-dimensional motions cannot sustain a dynamo [Zel'dovich's theorem, JETP 1957]
  - A purely toroidal flow cannot sustain a dynamo
  - $\mathbf{B}(x, y, t)$ cannot be a dynamo field
- Dynamos are only possible in "complex" geometries or flows
  - An extra burden for both theory and numerics
  - A popular "minimal" configuration is 2.5D (or 2D-3C)
    - $\mathbf{u}(x, y, t)$ with all three components non-vanishing

$$\mathbf{B}(x, y, z, t) = \mathcal{R}\left\{\mathbf{b}(x, y, t)e^{ik_z z}\right\}$$

The fast dynamo paradigm

[Vainshtein & Zel'dovich, SPU, 1972]

- Chaotic stretching, twisting, folding and merging of field lines
  - For small diffusion, field doubles at each "iteration" (characteristic time)
  - Exponential growth with "ideal" growth rate $\gamma_\infty = \ln 2$ ~ stretching rate



Stretch

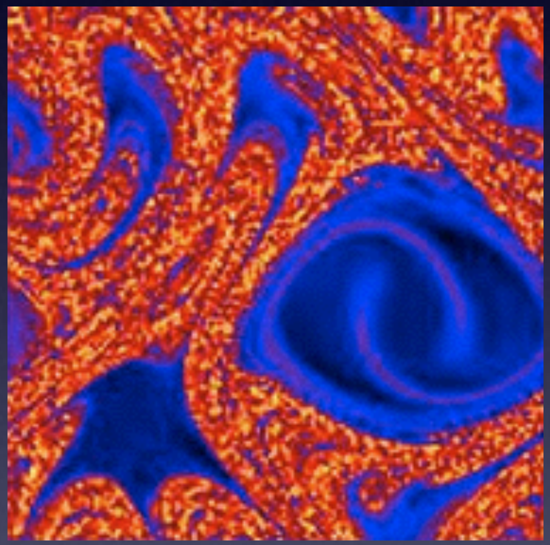
Twist

Fold

Merge
(requires a tiny bit
of magnetic diffusion !)

3D essential !

[adapted from Brandenburg &
Subramanian, Phys. Rep. 2005]



Lyapunov exponents of
Galloway-Proctor flow
[credits F. Cattaneo]

Small-scale dynamo theory

Numerical evidence

- Homogeneous, isotropic, non-helical, incompressible, 3D turbulent flow of conducting fluid is a small-scale dynamo

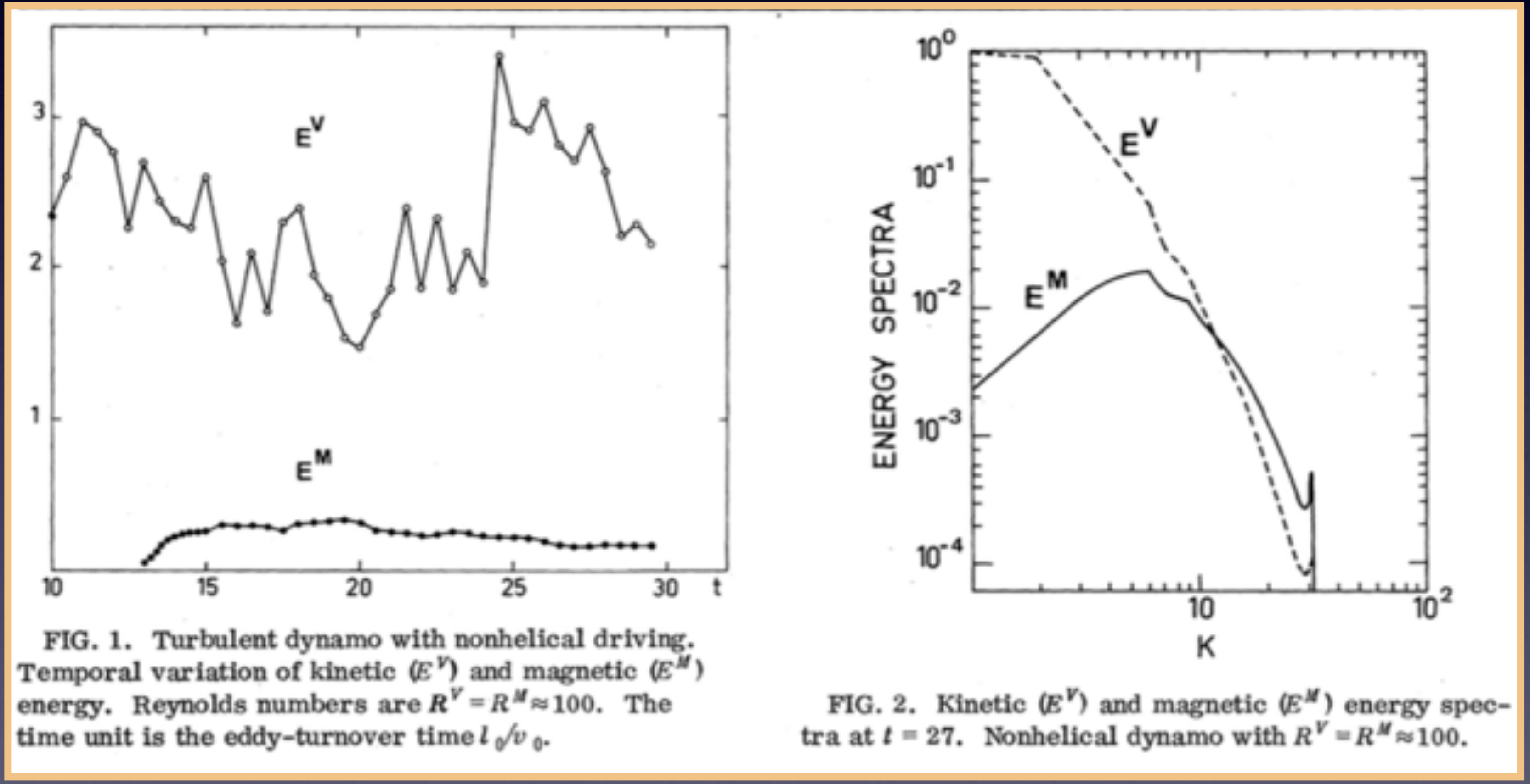


FIG. 1. Turbulent dynamo with nonhelical driving. Temporal variation of kinetic (E^V) and magnetic (E^M) energy. Reynolds numbers are $R^V = R^M \approx 100$. The time unit is the eddy-turnover time l_0/v_0.

FIG. 2. Kinetic (E^V) and magnetic (E^M) energy spectra at $t = 27$. Nonhelical dynamo with $R^V = R^M \approx 100$.

64x64x64 spectral DNS simulations at Pm=1

[Meneguzzi, Frisch, Pouquet, PRL, 1981]

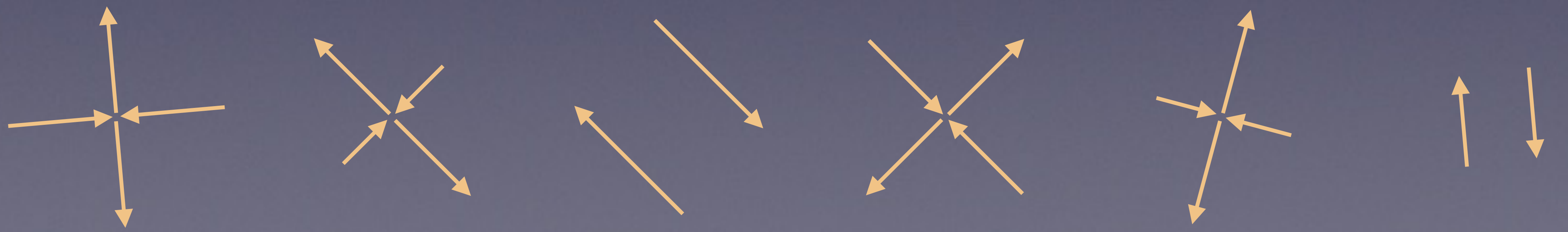
Zel’dovich phenomenology

[Zel’dovich et al., JFM 144, 1 (1984)]

- Consider incompressible, kinematic dynamo problem

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B} \qquad \nabla\cdot\mathbf{B} = 0$$

- Assume that $\mathbf{B}(0,\mathbf{r}) = \mathbf{B}_0(\mathbf{r})$
  - has finite total, energy, no singularity
  - $\lim_{r\to\infty} \mathbf{B}_0(\mathbf{r}) = 0$
- Take simplest possible model of time-evolving “smooth” velocity field
  - Random linear shear: $\mathbf{u} = \mathbf{C}\mathbf{r}$ $\quad \mathrm{Tr}\,\mathrm{C} = 0$ [incompressible]



[think of this as being 3D]

Stretching and squeezing

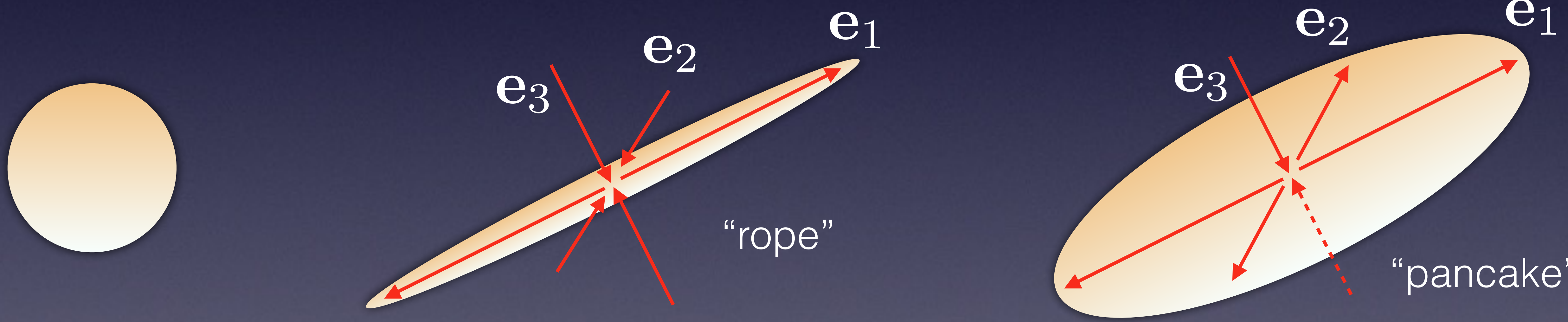
- Evolution of vector connecting 2 fluid particles: $\frac{d\delta r_i}{dt} = \mathsf{C}_{\mathsf{ik}}\delta r_k$
- Consider constant $\mathsf{C} = \mathrm{diag}(c_1, c_2, c_3)$

$$c_1 + c_2 + c_3 = 0$$

  - Exponential stretching along first axis

$$c_1 > 0 > c_2 > c_3$$

$$c_1 > c_2 > 0 > c_3$$



- In ideal MHD, we thus expect $B^2 \sim \exp(2c_1 t)$
  - However, perpendicular squeezing implies that even a tiny magnetic diffusion matters…is growth still possible in that case ?

Magnetic field evolution

- Decompose $\mathbf{B}(t,\mathbf{r}) = \int \mathbf{b}(t,\mathbf{k}_0)\exp\left(i\mathbf{k}(t)\cdot\mathbf{r}\right)d^3\mathbf{k}_0$

$$\frac{d\mathbf{b}}{dt} = \mathsf{C}\mathbf{b} - \eta \mathsf{k}^2\mathbf{b} \qquad \frac{d\mathbf{k}}{dt} = -\mathsf{C}^{\mathsf{T}}\mathbf{k} \qquad \mathbf{k}\cdot\mathbf{b} = 0$$

- Diffusive part of evolution ~ $\exp\left(-\eta\int_0^t k^2(s)ds\right)$
  - super-exponential decay of most Fourier modes because

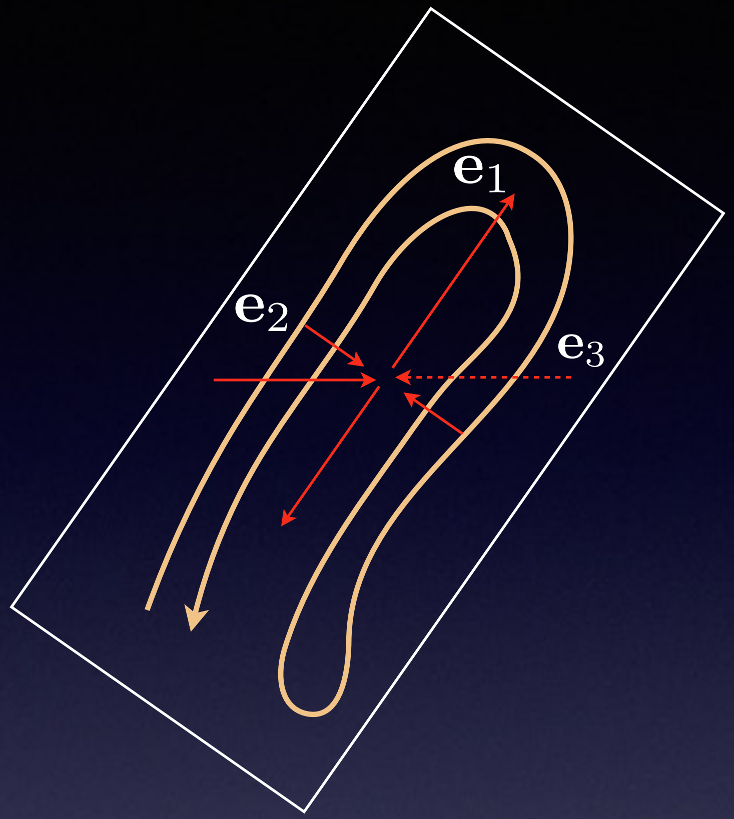
$$k_3 \sim k_{03}\exp(|c_3|t)$$

  - survivors live in an exponentially narrow cone of modes such that

$$\eta\int_0^t k^2(s)ds = O(1)$$

    - rope case: $k_{02} \sim \exp(-|c_2|t) \qquad k_{03} \sim \exp(-|c_3|t)$

Magnetic field evolution (ropes)



- Surviving modes at time t have an initial field
  - $b_1(0,\mathbf{k}_0) \sim b_2(0,\mathbf{k}_0)k_{02}/k_{01} \sim \exp(-|c_2|t)$
  - This field is stretched along the first axis, so

$$\mathbf{b}(t,\mathbf{k}_0) \sim \exp\left(c_1 t\right)\exp\left(-|c_2|t\right)$$

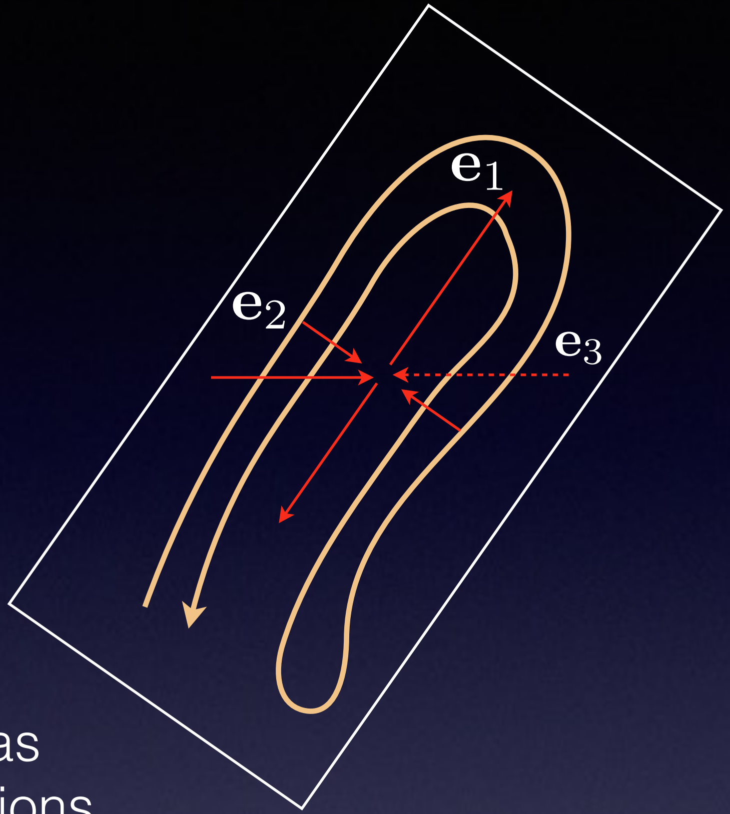
- Now, estimate the magnetic field in physical space

$$\mathbf{B}(t,\mathbf{r}) \sim \int \underset{\sim \exp\left[(c_1-|c_2|)t\right]}{\mathbf{B}_k}\, d^3\underset{\sim \exp\left[(-|c_2|-|c_3|)t\right]}{\mathbf{k}_0} \sim \exp(-|c_2|t)$$

Magnetic field stretches into an asymptotically-decaying rope

Magnetic energy evolution (ropes)

- What about magnetic energy ?



$$E_\mathrm{m} = \int \mathbf{B}^2(t,\mathbf{r})d^3\mathbf{r}$$

$$B^2 \sim \exp\left(-2|c_2|t\right)$$

$$\mathrm{Volume} \sim \exp(c_1 t)$$

Important: no shrinking along axis 2 and 3 as diffusion sets a minimum scale in these directions

$$E_\mathrm{m} \sim \exp\left[(c_1 - 2|c_2|)t\right] \sim \exp\left[(|c_3| - |c_2|)t\right]_{\text{(3D)}}$$

Total magnetic energy grows ! (in 3D)

Volume occupied by the magnetic field grows faster than field decays pointwise

- Similar conclusions apply in the pancake case, but $E_\mathrm{m} \sim \exp\left[(c_1 - c_2)t\right]$

Generalization to random, time-dependent shear

- Renovate shear flow every time-interval τ

- Succession of random area-preserving stretches and squeezes
  - Introduce the matrix $\mathsf{T}_t \equiv \mathsf{T}(t_0, t)$ such that $\mathbf{k}(t, \mathbf{k}_0) = \mathsf{T}_t \mathbf{k}_0$
  - Volterra multiplicative integral form: $\mathsf{T}_t = \prod_{s=0}^{t} \left[\mathbb{1} - \mathsf{C}^T(s) ds\right]$

Product of unimodular random matrices

- Formal solution

$$\mathbf{B}(t, \mathbf{r}) = \int \exp\left[i\mathsf{T}_t\left(\mathbf{k}_0 \cdot \mathbf{r}\right) - \eta \int_0^t (\mathsf{T}_s \mathbf{k}_0)^2 ds\right] (\mathsf{T}_t^T)^{-1} \mathbf{b}(0, \mathbf{k}_0) d^3\mathbf{k}_0$$

  - Hard work: calculate the properties of the multiplicative integral !

Lyapunov basis of random shear flow

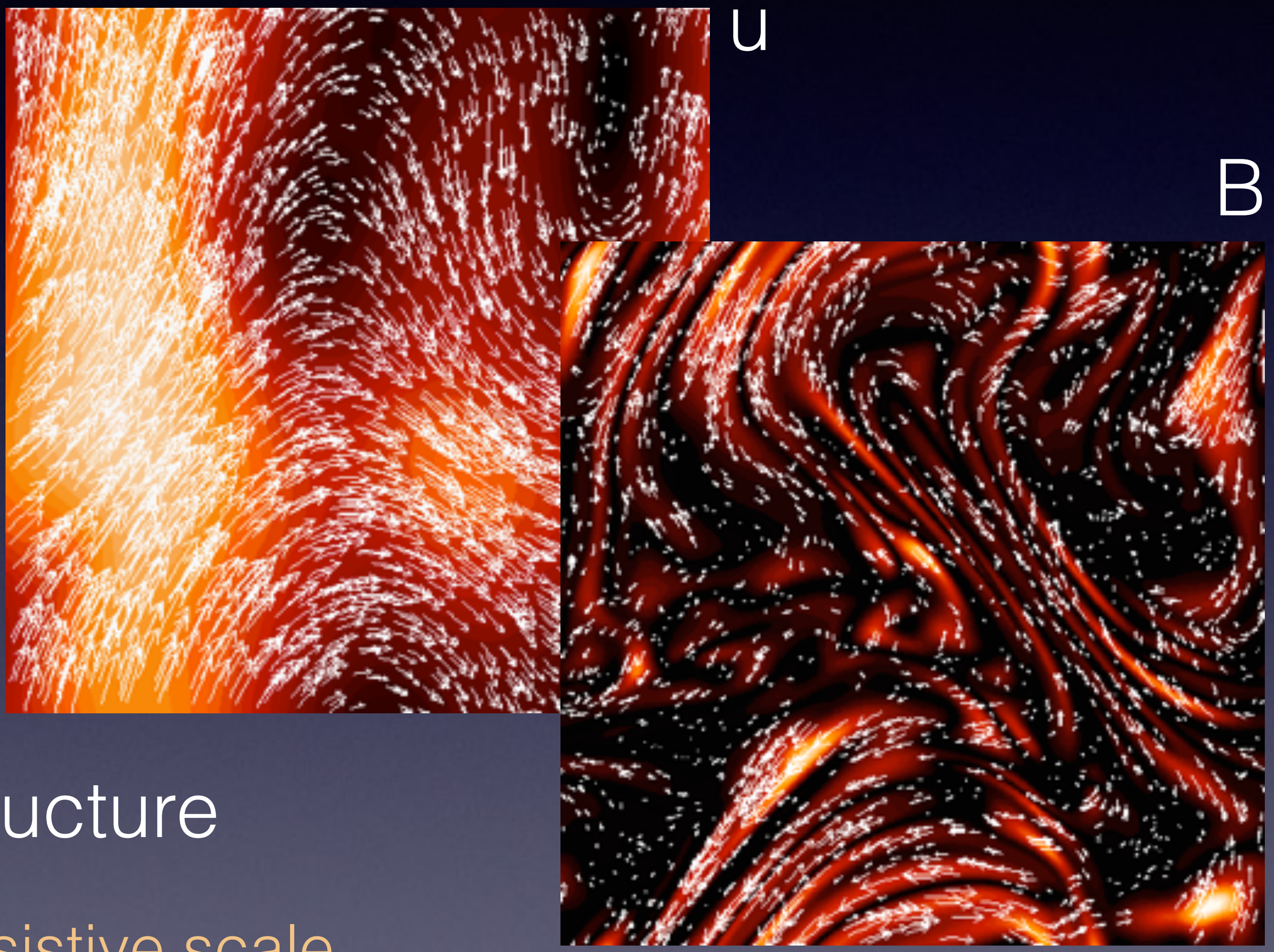
- Zel'dovich showed that the cumulative effects of any random sequence of shears can be reduced to diagonal form
  - In particular there is always a net positive "stretching" Lyapunov exponent

$$\lim_{n\to\infty} \frac{1}{n\tau} \ln k(n\tau, \mathbf{k_0}) \equiv \gamma_1 > 0$$

- The underlying Lyapunov basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$
  - is a function of the full random sequence, but is independent of time
  - "cristallizes" exponentially fast in time (exponents converge as 1/t)
- The problem reduces to that described earlier
  - Magnetic energy growth is possible in a smooth, 3D chaotic velocity field in the presence of magnetic diffusion

Small-scale dynamo fields at Pm ≥ 1

- Pm=Rm=1250, Re=1 [from Schekochihin et al., ApJ 2004]



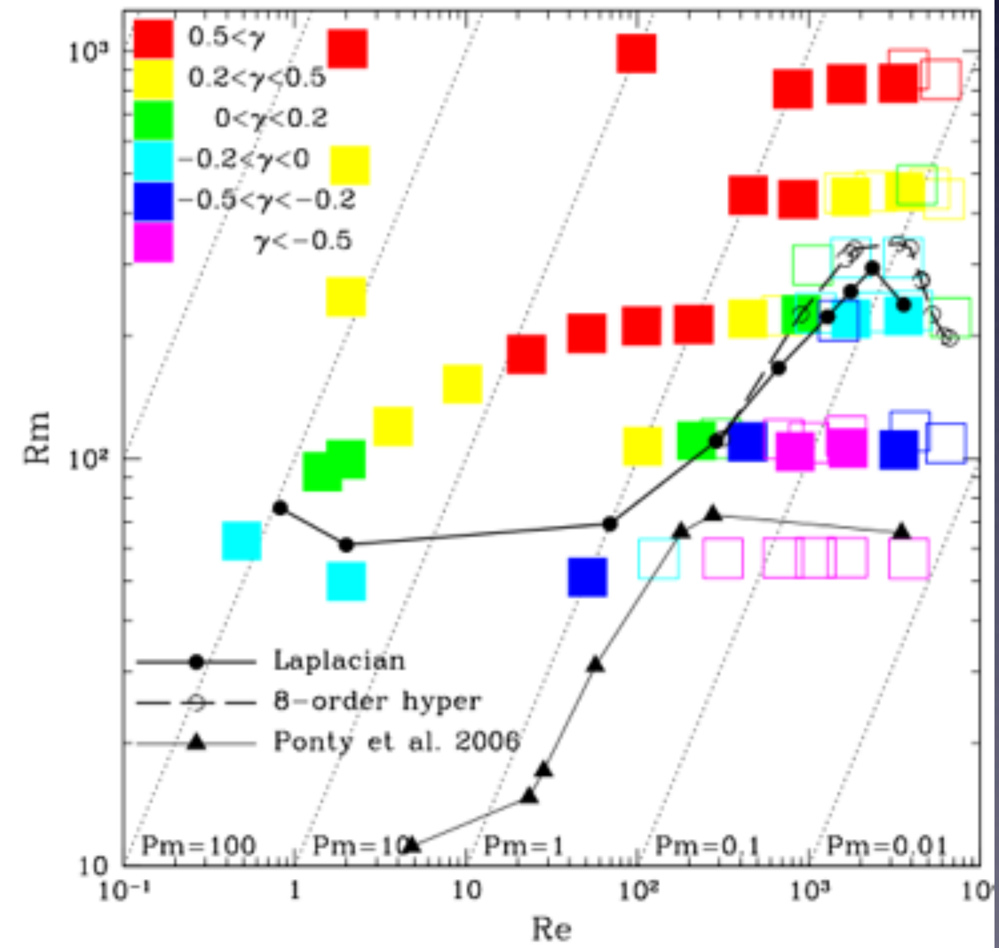
$$\ell_\eta \sim \ell_\nu \mathrm{Pm}^{-1/2}$$

Critical Rm ~ 60

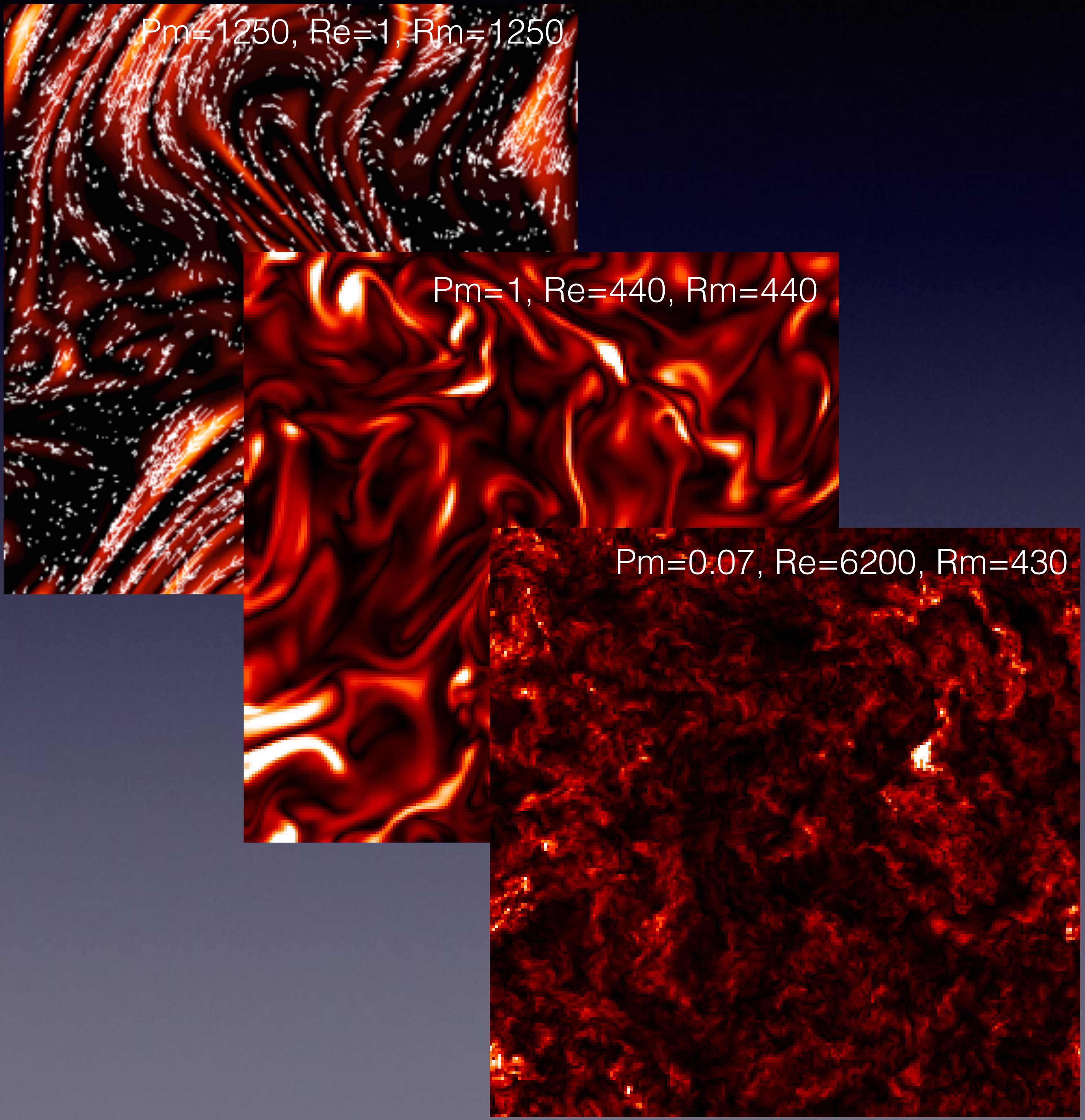
- Folded field structure
  - Reversals at resistive scale
  - Folds coherent over flow scale
  - Field strength and curvature anticorrelated

Small-scale dynamo at low Pm

- Yes, but much harder
  - Critical Rm~200
  - More complicated than Zel'dovich picture



[Iskakov et al., PRL 2007]



Introduction to Kazantsev-Kraichnan

- Consider again the following kinematic dynamo problem:

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B} \qquad \nabla\cdot\mathbf{B} = 0$$

- This problem can be solved analytically if u is
  - a random Gaussian process with no memory (zero-correlation time)
  - The so-called Kraichnan ensemble
- Obviously, not your usual turbulent flow, but still…
  - Very useful to understand the properties of small-scale dynamo modes
- Originally solved by Kazantsev [JETP, 1968]

[and further explored by Zel'dovich, Ruzmaikin, Sokoloff, Vainshtein,
Kitchatinov, Vergassola, Vincenzi, Subramanian, Boldyrev, Schekochihin etc.]

Basic assumptions on the velocity

- 3D, statistically steady, homogeneous

$$\langle u^i(\mathbf{x},t)u^j(\mathbf{x}',t')\rangle = R^{ij}(\mathbf{x}-\mathbf{x}',t-t')$$

- Gaussian
  - pdf $P[\mathbf{u}] = C\exp\left[-\frac{1}{2}\int dt\int dt'\int d^3\mathbf{x}\int d^3\mathbf{x}'D_{ij}(t-t',\mathbf{x}-\mathbf{x}')u^i(t,\mathbf{x})u^j(t',\mathbf{x}')\right]$
  - Covariance matrix $\int d\tau\int d^3\mathbf{y}D_{ik}(t-\tau,\mathbf{x}-\mathbf{y})R^{kj}(\tau-t',\mathbf{y}-\mathbf{x}') = \delta_i^j\delta(t-t')\delta(\mathbf{x}-\mathbf{x}')$
- Vanishing correlation time: $\langle u^i(\mathbf{x},t)u^j(\mathbf{x}',t')\rangle = \kappa^{ij}(\mathbf{r})\delta(t-t')$
- Isotropic and non-helical: $\kappa^{ij}(\mathbf{r}) = \kappa_N(r)\left(\delta^{ij}-\frac{r^ir^j}{r^2}\right)+\kappa_L(r)\frac{r^ir^j}{r^2}$
- Incompressible: $\kappa_N = \kappa_L + (r\kappa_L')/2$

$$\mathbf{r} = \mathbf{x}-\mathbf{x}'$$

Equation for the magnetic correlation

- Goal: derive a closed equation for the two-point, single time magnetic correlator [or equivalently magnetic spectrum]

$$\big\langle B^i(\mathbf{x},t)B^j(\mathbf{x}',t)\big) = H^{ij}(\mathbf{x}-\mathbf{x}',t)$$

$$H^{ij}(\mathbf{x}-\mathbf{x}',t) = H_N(r,t)\left(\delta^{ij} - \frac{r^i r^j}{r^2}\right) + H_L(r,t)\frac{r^i r^j}{r^2}$$

$$H_N = H_L + (rH_L')/2$$

- Induction equation at (x,t) and (x’,t) gives

$$\begin{aligned} \frac{\partial H^{ij}}{\partial t} &= \frac{\partial}{\partial x'^k}\left(\langle B^i(\mathbf{x},t)B^k(\mathbf{x}',t)u^j(\mathbf{x}',t)\rangle - \langle B^i(\mathbf{x},t)B^j(\mathbf{x}',t)u^k(\mathbf{x}',t)\rangle\right) \\ &+ \frac{\partial}{\partial x^k}\left(\langle B^k(\mathbf{x},t)B^j(\mathbf{x}',t)u^i(\mathbf{x},t)\rangle - \langle B^i(\mathbf{x},t)B^j(\mathbf{x}',t)u^k(\mathbf{x},t)\rangle\right) \\ &+ \eta\left(\frac{\partial}{\partial x^k} + \frac{\partial}{\partial x'^k}\right)H^{ij} \end{aligned}$$

Second order moment

Third order moments

$$\frac{\partial}{\partial x'^k}\langle\cdot\rangle = -\frac{\partial}{\partial x^k}\langle\cdot\rangle = \frac{\partial}{\partial r^k}\langle\cdot\rangle$$

[statistical homogeneity]

Closure procedure in a nutshell

- Velocity field is Gaussian, so we use the Furutsu-Novikov formula [Gaussian integration]

$$\langle u^i(\mathbf{x},t)F[\mathbf{u}]\rangle = \int dt'' \int d^3\mathbf{x}'' \,\langle u^i(\mathbf{x},t)u^\ell(\mathbf{x}'',t'')\rangle \left\langle \frac{\delta F[\mathbf{u}]}{\delta u^\ell(\mathbf{x}'',t'')}\right\rangle$$

  - Reduction into integrals of products of second order moments only, e.g.

$$\langle u^i(\mathbf{x},t)B^k(\mathbf{x},t)B^j(\mathbf{x}',t)\rangle = \int_0^t dt'' \int d^3\mathbf{x}'' \,\langle u^i(\mathbf{x},t)u^\ell(\mathbf{x}'',t)\rangle \left\langle \frac{\delta\left[B^k(\mathbf{x},t)B^j(\mathbf{x}',t)\right]}{\delta u^\ell(\mathbf{x}'',t'')}\right\rangle$$

    - The time integral can be done thanks to vanishing correlation time assumption $\langle u^i(\mathbf{x},t)u^\ell(\mathbf{x}'',t'')\rangle = \kappa^{i\ell}(\mathbf{x}-\mathbf{x}'')\delta(t-t'')$

    - Functional derivatives are computed from formal solutions of the induction equation, e.g.

$$B^k(\mathbf{x},t) = \int^t dt' \left[B^m(\mathbf{x},t')\frac{\partial u^k(\mathbf{x},t')}{\partial x^m} - u^m(\mathbf{x},t')\frac{\partial B^k(\mathbf{x},t')}{\partial x^m} + \eta\Delta B^k(\mathbf{x},t')\right]$$

    - The space integrals become easy, as the functional derivatives introduce $\delta(\mathbf{x}'-\mathbf{x}'')$ and $\delta(\mathbf{x}-\mathbf{x}'')$

The closed equation

- Using the appropriate projection operators, the problem reduces to a closed equation for the scalar function $H_L(r,t)$

$$\frac{\partial H_L}{\partial t} = \kappa H_L'' + \left(\frac{4}{r}\kappa + \kappa'\right) H_L' + \left(\kappa'' + \frac{4}{r}\kappa'\right) H_L$$

$$\kappa(r) = 2\eta + \kappa_L(0) - \kappa_L(r)$$ “Turbulent diffusivity” (twice)

- Schrödinger equation with imaginary time
  - Change variables: $H_L(r,t) = \psi(r,t) r^{-2} \kappa(r)^{-1/2}$

$$\frac{\partial \psi}{\partial t} = \kappa(r)\psi'' - V(r)\psi$$

  - Wave function of quantum particle of variable $m(r) = \frac{1}{2\kappa(r)}$ in potential

$$V(r) = \frac{2}{r^2}\kappa(r) - \frac{1}{2}\kappa''(r) - \frac{2}{r}\kappa'(r) - \frac{\kappa'(r)^2}{4\kappa(r)}$$

Solutions

$$\frac{\partial \psi}{\partial t} = \frac{\psi''}{2m(r)} - V(r)\psi$$

$$m(r) = [2\kappa(r)]^{-1}$$

$$\kappa(r) = 2\eta + \kappa_L(0) - \kappa_L(r)$$

$$V(r) = \frac{2}{r^2}\kappa(r) - \frac{1}{2}\kappa''(r) - \frac{2}{r}\kappa'(r) - \frac{\kappa'(r)^2}{4\kappa(r)}$$

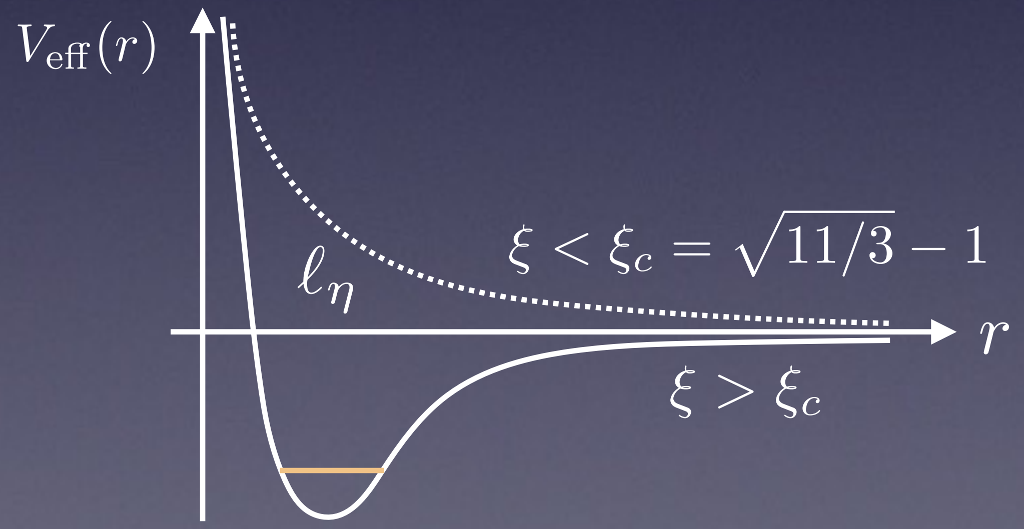
- Look for solutions of the form $\psi = \psi_E(r)e^{-Et}$
  - Growing dynamo modes correspond to discrete bound states: E<0
    - Variational result: $E = \dfrac{\int 2mV{\psi_E}^2 dr + \int {\psi_E'}^2 dr}{\int 2m{\psi_E}^2 dr}$
  - To determine whether dynamo takes place, we can equivalently solve

$$\psi_E'' + [E - V_{\text{eff}}(r)]\,\psi_E = 0 \qquad V_{\text{eff}}(r) = V(r)/\kappa(r)$$

    - The ground state describes the long-time asymptotics

Different regimes

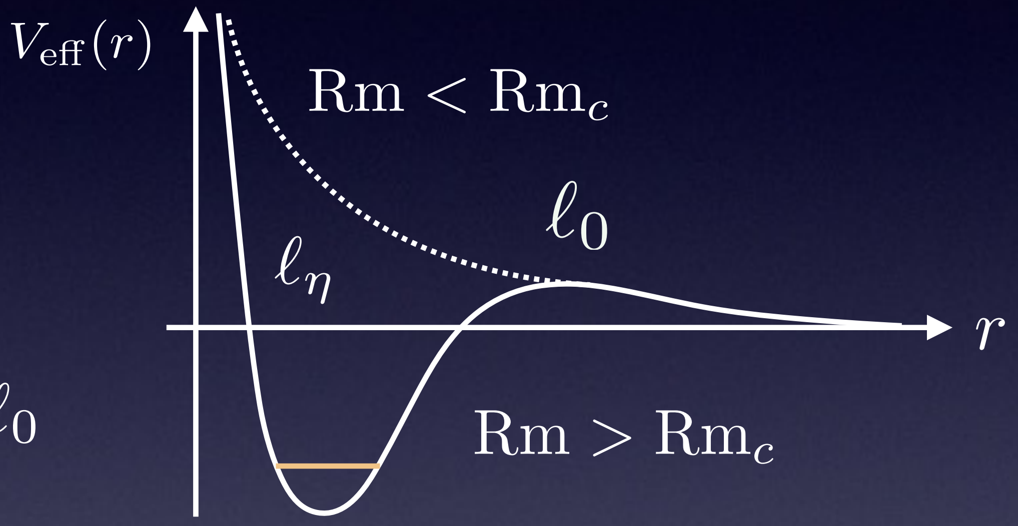
- Recall $\langle u^i(\mathbf{x},t)u^\ell(\mathbf{x}'',t'')\rangle = \kappa^{i\ell}(\mathbf{x}-\mathbf{x}'')\delta(t-t'')$
  - So $\kappa(r) \sim \delta u(r)^2\tau(r) \;\; \sim r\delta u(r)$ is akin to a turbulent diffusivity
- Consider the scaling law $\kappa_L(0)-\kappa_L(r) \sim r^\xi$ Roughness exponent
  - Smooth flow: $\delta u \sim r \quad \Rightarrow \quad \xi = 2$ [“large Pm”]
  - “Kolmogorov” turbulence: $\delta u \sim r^{1/3} \quad \Rightarrow \quad \xi = 1 + 1/3 = 4/3$ [“low Pm”]
- Potential as a function of ξ
  - $V_{\text{eff}}(r) = 2/r^2, \quad r \ll \ell_\eta$
  - $V_{\text{eff}}(r) = (2 - \frac{3}{2}\xi - \frac{3}{4}\xi^2)/r^2, \quad r \gg \ell_\eta$



- Growing bound modes for $\xi > 1$
  - includes both Pm >> 1 and Pm << 1 [“K41”] regimes

Finite Rm effects

- Introduce an integral scale ℓ_0 beyond which u decorrelates
  - Finite scale separation ℓ_0/ℓ_η



- Potential asymptotics
  - $V_{\rm eff}(r) = 2/r^2, \quad r \ll \ell_\eta$
  - $V_{\rm eff}(r) = (2 - \frac{3}{2}\xi - \frac{3}{4}\xi^2)/r^2, \quad \ell_\eta \ll r \ll \ell_0$
  - $V_{\rm eff}(r) = 2/r^2, \quad \ell_0 \ll r$ repulsive
- Existence of potential well depends on how large ℓ_0/ℓ_η is
  - Critical Rm for the dynamo below which diffusion wins over stretching
  - Applies to all dynamo velocity fields ($\xi > 1$) and Prandtl numbers

A few interesting results at large Pm

- Consider the so called Batchelor regime $\ell_\eta \ll \ell_\nu$
  - The magnetic field is stretched and advected by a viscous flow
  - The velocity field is smooth: $\kappa^{ij}(r) = \kappa_0 \delta^{ij} - \kappa_2 \frac{r^2}{2}\left(\delta^{ij} - \frac{1}{2}\frac{r^i r^j}{r^2}\right) + \cdots$
- Spectral view at scales much smaller than the viscous scale
  - Work under Kazantsev-Kraichnan assumptions
  - Fokker-Planck type equation for the magnetic spectrum M(k)

$$\frac{\partial M}{\partial t} = \frac{\gamma}{5}\left(k^2 \frac{\partial^2 M}{\partial k^2} - 2\frac{\partial M}{\partial k} + 6M\right) - 2\eta k^2 M$$

  - Typical growth rate of the order of the shearing rate at viscous scales

$$\gamma = \frac{5}{4}\kappa_2 = \frac{5}{2}|\kappa_L''(0)| \sim \delta u_\nu / \ell_\nu$$

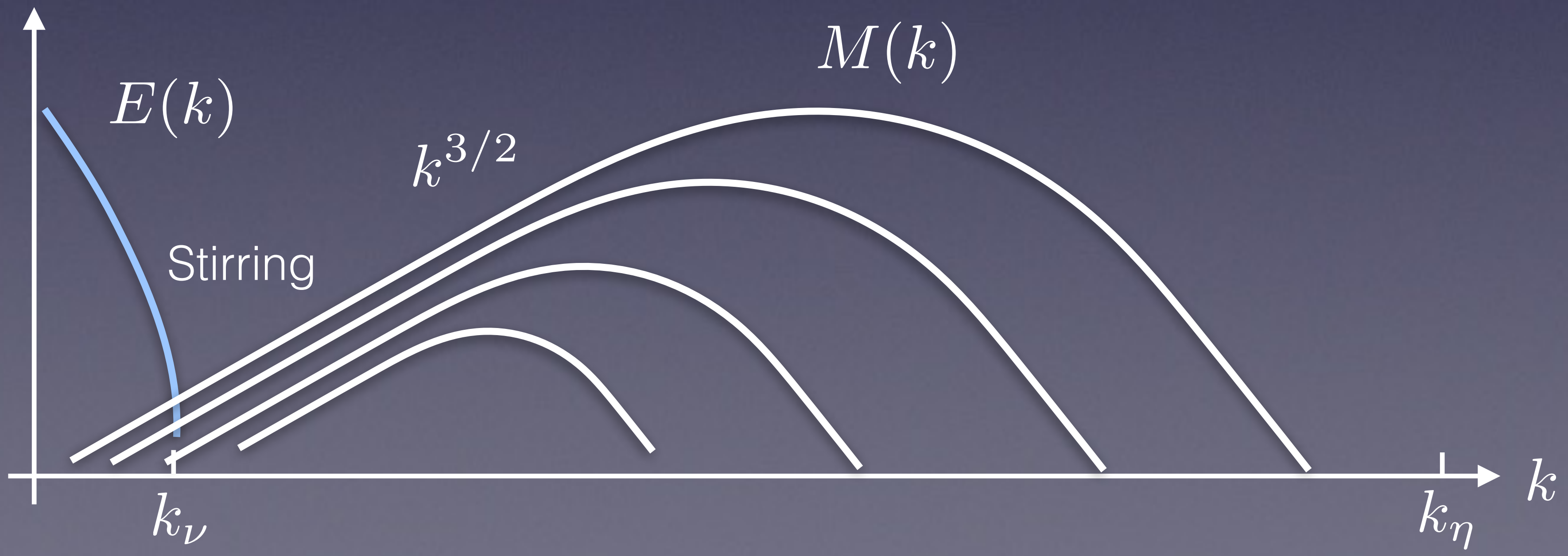
[Kazantsev JETP 1968;
Kulsrud and Anderson, ApJ 1992;
Schekochihin et al., ApJ 2002]

Diffusion-free regime

- Magnetic diffusion negligible if magnetic field only has $k \ll k_\eta$
  - If we excite a given k₀ initially, the spectrum spreads towards small-scale

$$M(k,t) \propto e^{3/4\gamma t} \left(\frac{k}{k_0}\right)^{3/2} \sqrt{\frac{5}{4\pi\gamma t}} \exp\left[-\frac{5\,\ln^2(k/k_0)}{4\gamma t}\right]$$

  - The energy of each mode grows at rate $3\gamma/4$
  - Total energy grows at rate 2γ as the number of excited mode also grows
  - The magnetic field develops the so-called $k^{3/2}$ Kazantsev spectrum



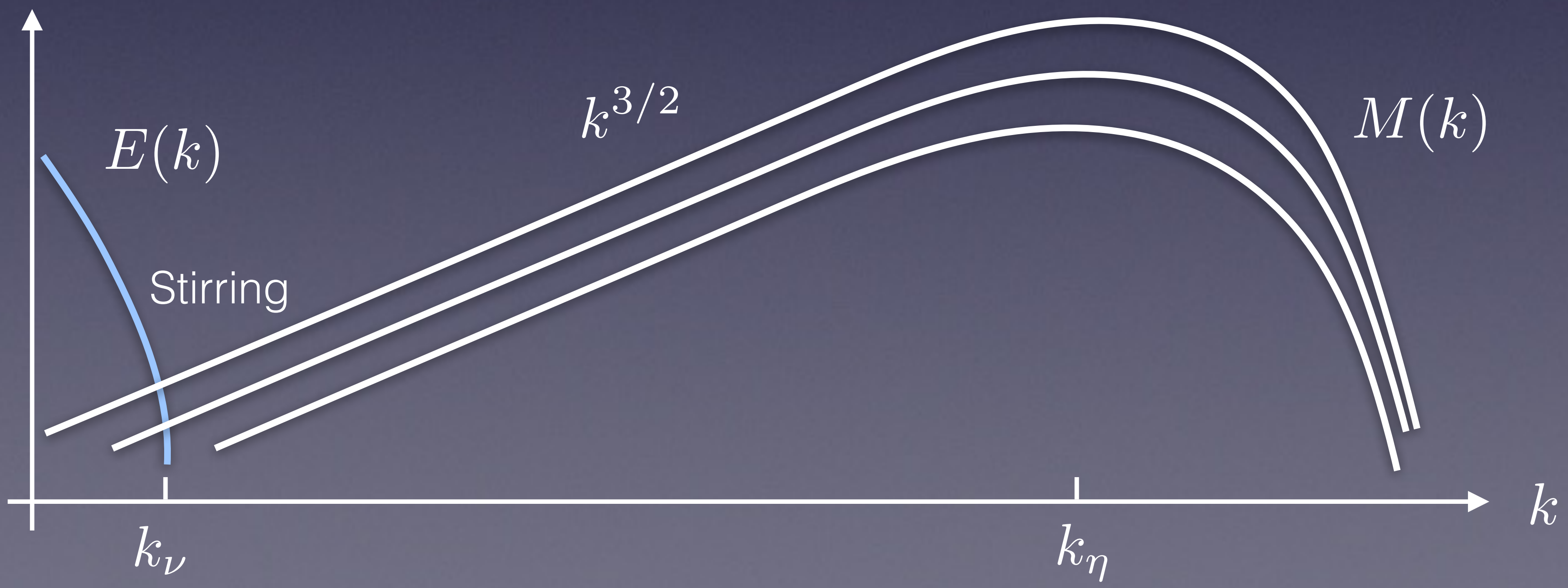
Resistive regime

- After the spectrums hits $k \sim k_\eta$, the long-time asymptotics is

$$M(k) \propto k^{3/2} K_0\left(\frac{k}{k_\eta}\right) e^{3/4\gamma t} \qquad k_\eta = \sqrt{\gamma/(10\eta)} \sim \mathrm{Pm}^{1/2} k_\nu$$

  - The spectrum peaks at the resistive scale [falls off exponentially beyond]
  - The asymptotic total energy growth rate is now also $3\gamma/4$

    [Hidden subtleties: weak dependence of the actual growth rate on boundary condition at small k]



Magnetic pdf in the diffusion-free regime

- Introduce the characteristic function

$$Z(\mu,t) = \langle \exp\left[i\mu_i B^i(\mathbf{x},t)\right]\rangle = \left\langle \tilde{Z} \right\rangle$$

  - This function is the Fourier transform in B of the p.d.f.

$$Z(\mu,t) = \int P\left[\mathbf{B}\right] \exp\left[i\mu_i B^i(\mathbf{x},t)\right] d^3\mathbf{B}$$

  - Use the ideal induction equation in the Lagrangian frame

$$\frac{\partial Z}{\partial t} = \mu_i \frac{\partial}{\partial \mu_k} \left\langle \frac{\partial u^i}{\partial x^k} \tilde{Z} \right\rangle$$

  - The equation is closed using the same tricks as before

$$\frac{\partial Z}{\partial t} = \frac{\kappa_2}{2} T^{ij}_{k\ell}\, \mu_i \frac{\partial}{\partial \mu_k} \mu_j \frac{\partial}{\partial \mu_\ell} Z$$

$$T^{ij}_{k\ell} = -\frac{1}{\kappa_2} \frac{\partial^2 \kappa^{ij}(\mathbf{r})}{\partial r^k \partial r^\ell} = \delta^{ij}\delta_{k\ell} - \frac{1}{4}\left(\delta^i_k \delta^j_\ell + \delta^i_\ell \delta^j_k\right)$$

Strain correlator [3D, incompressible]

Magnetic pdf in the diffusion-free regime

- Fokker-Planck equation for the pdf
  - Fourier transform back in μ, using $\mu_i \frac{\partial}{\partial \mu_k} () = \frac{\partial}{\partial B^i} \left[B^k () \right]$

$$\frac{\partial}{\partial t} P\left[\mathbf{B}\right] = \frac{\kappa_2}{2} T^{ij}_{k\ell} B^k \frac{\partial}{\partial B^i} B^\ell \frac{\partial}{\partial B^j} P\left[\mathbf{B}\right]$$

  - Simplifies in the isotropic case as 1D diffusion equation with drift

$$\frac{\partial}{\partial t} P\left[B\right] = \frac{\kappa_2}{4} \frac{1}{B^2} \frac{\partial}{\partial B} B^4 \frac{\partial}{\partial B} P\left[B\right]$$

pdf defined as

$$\langle B^n(t) \rangle = 4\pi \int_0^\infty dB B^{2+n} P\left[B\right]$$

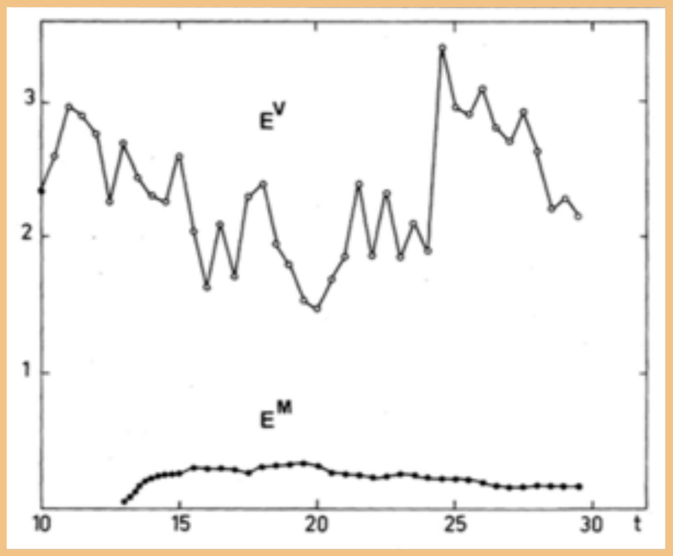
- Lognormal solution

$$P[B](t) = \frac{1}{\sqrt{\pi \kappa_2 t}} \int_0^\infty \frac{dB'}{B'} P_0\left[B'\right] \exp\left(- \frac{\left[\ln(B/B') + (3/4)\kappa_2 t\right]^2}{\kappa_2 t} \right)$$

  - The magnetic field is strongly intermittent
  - Magnetic moments grow as $\langle B^n(t) \rangle \propto \exp\left[\frac{1}{4} n(n+3)\kappa_2 t \right]$

Saturation of small-scale dynamo

- As B gets large-enough, Lorentz force saturates dynamo
  - What is “large-enough “?
  - How does it work ?



[Meneguzzi et al., PRL 1981]

- Historical ideas
  - Batchelor argument [PRSL,1950]:
    - magnetic field is similar to hydrodynamic vorticity
    - should peak at viscous scale, hence saturation for $B^2 \sim \delta u_\nu^2$

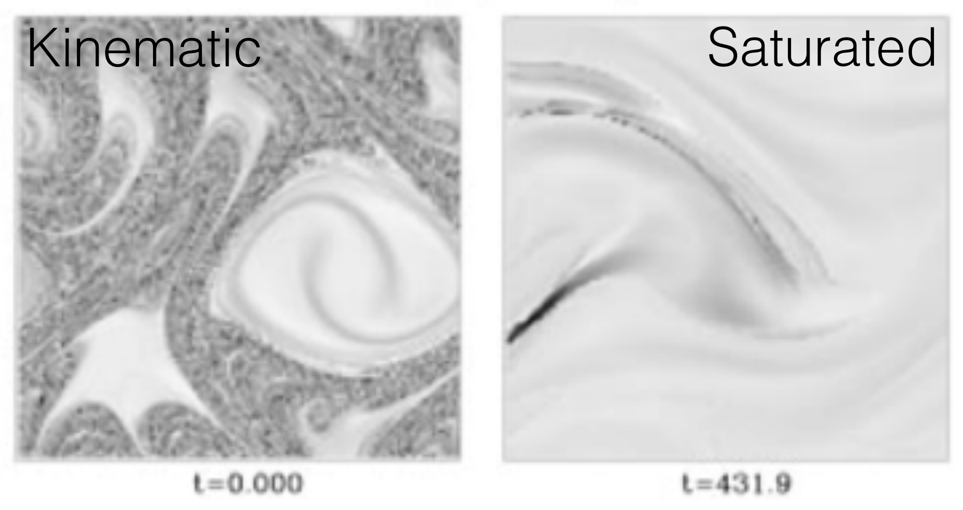
$$\left\langle B^2 \right\rangle \sim \mathrm{Re}^{-1/2} \left\langle u^2 \right\rangle$$

Sub-equipartition unless Re=1

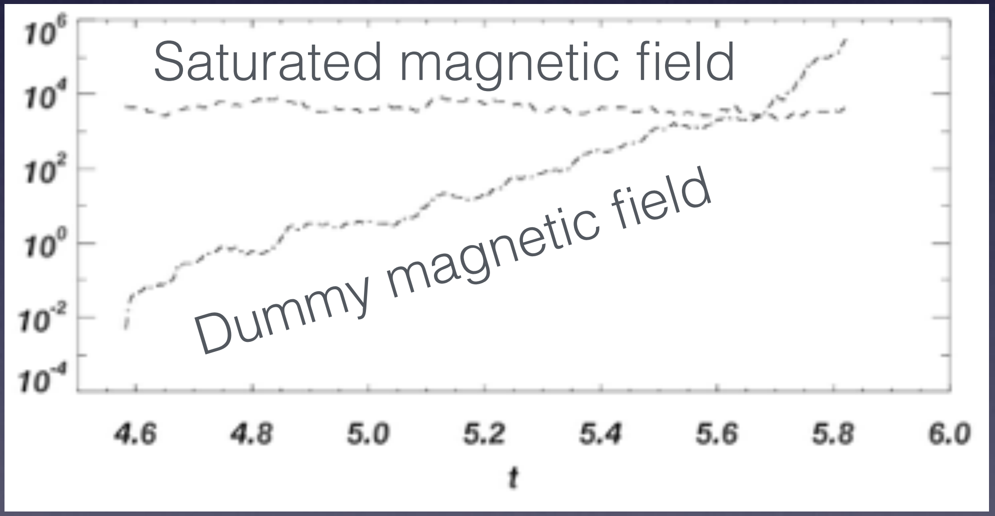
  - Schlüter-Biermann argument [Z. Naturforsch.,1950]:
    - equipartition at all scales $\left\langle B^2 \right\rangle \sim \left\langle u^2 \right\rangle$

Saturation phenomenology

- Geometric structure and orientation of the field matters
  - Magnetic tension $\mathbf{B}\cdot\nabla\mathbf{B}$ encodes magnetic curvature
  - Reduction of stretching Lyapunov exponents
  - A field realization can only saturate itself



[Cattaneo et al., PRL 1996]



[Cattaneo & Tobias, JFM 2008]

- Saturation at low Pm
  - Pretty much *Terra incognita* (no published simulation)

Large Pm phenomenology

- Plausible (but not definitive) scenario from simulations
  [Schekochihin et al., ApJ 2002, 2004]
  - Lorentz force first suppresses stretching at viscous scales

$$\mathbf{B}\cdot\nabla\mathbf{B} \sim \mathbf{u}\cdot\nabla\mathbf{u} \sim \delta u_\nu^2/\ell_\nu$$

$$\sim B^2/\ell_\nu \quad \text{(folded structure)} \longrightarrow \left\langle B^2\right\rangle \sim \mathrm{Re}^{-1/2}\left\langle u^2\right\rangle$$

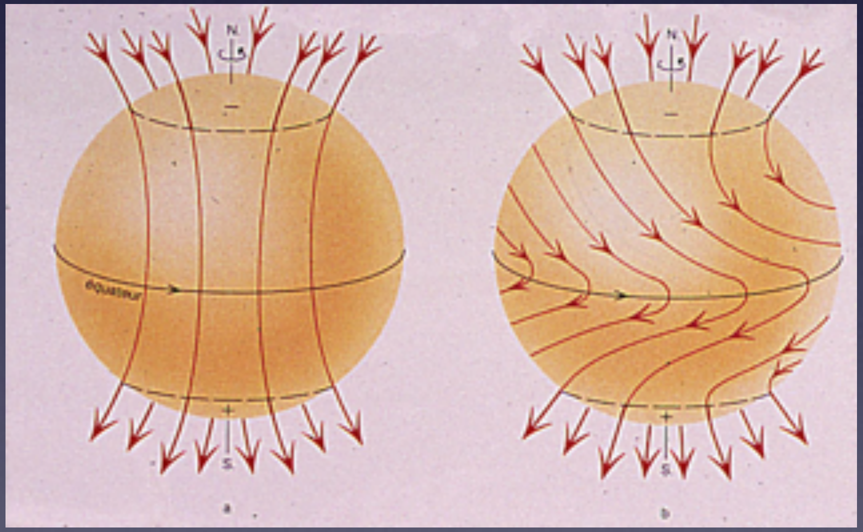
  - From there, slower, larger-scale eddies take over stretching
    - B keeps growing and acts on increasingly more energetic eddies…
      - Secular growth regime: $\left\langle B^2\right\rangle \sim \varepsilon t$
  - Final state: $\left\langle B^2\right\rangle \sim \left\langle u^2\right\rangle$ after “suppression” of full inertial range
    - “Isotropic MHD turbulence”, folded structure is preserved
- P[B] not log-normal anymore (likely exponential)

Large-scale dynamo theory

Differential rotation: the Omega effect

- Shearing of magnetic field by differential rotation (shear)
  - In polar geometry, consider the initial axisymmetric configuration
    - a purely poloidal magnetic field: $\mathbf{B}_{\rm pol} = B_r(r,z)\mathbf{e}_r + B_z(r,z)\mathbf{e}_z$
    - a toroidal, shearing velocity field (differential rotation): $\mathbf{u} = r\Omega(r,z)\mathbf{e}_\varphi$

$$\frac{\partial B_\varphi}{\partial t} = r(\mathbf{B}_{\rm pol}\cdot\nabla)\Omega + \eta\left(\Delta - \frac{1}{r^2}\right)B_\varphi$$

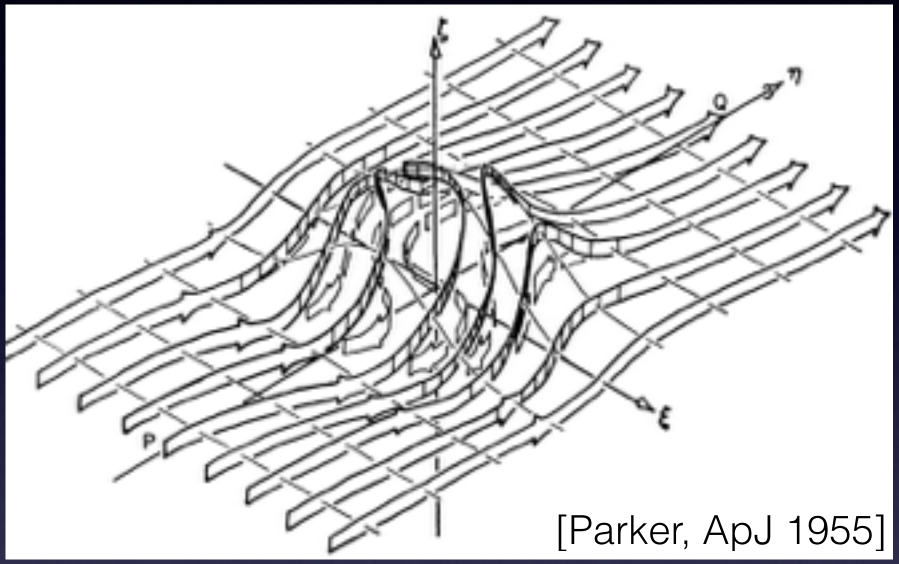


  - On short times, B_φ can grow linearly in time
  - Ultimately, diffusion always dominates

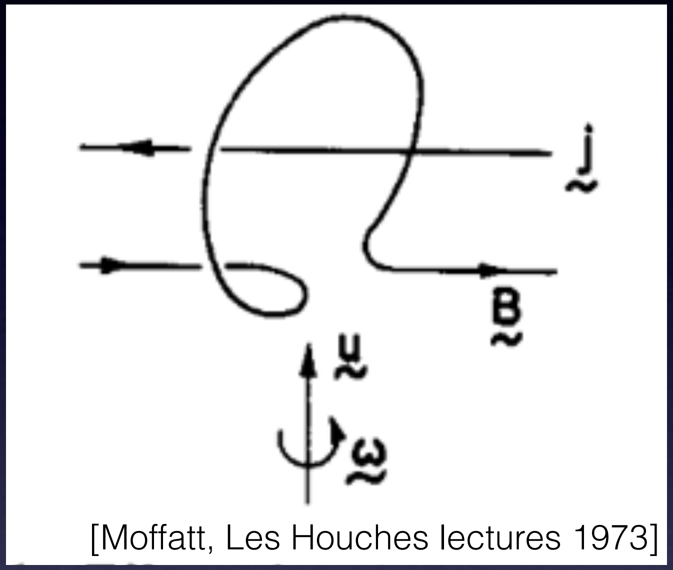
- This effect alone cannot not produce a dynamo (Cowling)
  - But it can transiently make strong toroidal field out of weak poloidal field

Turbulence: Parker’s mechanism

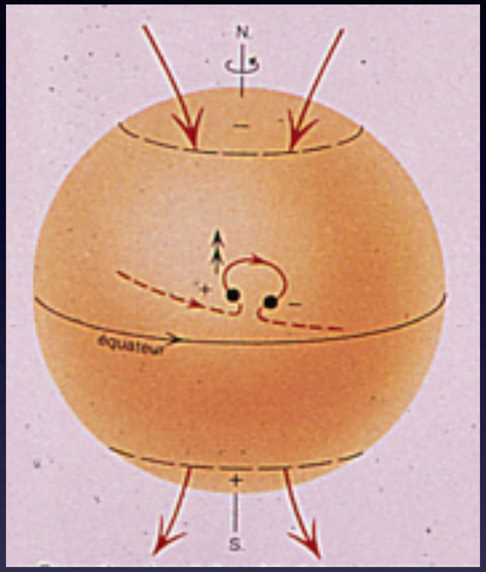
- Effect of a localized cyclonic swirl on a straight magnetic field



[Parker, ApJ 1955]



[Moffatt, Les Houches lectures 1973]



  - In polar geometry, this mechanism can produce axisymmetric poloidal field out of axisymmetric toroidal field — and the converse
    - Kinetic helicity in the swirl is essential

- This “alpha effect” can mediate statistical dynamo action
  - Ensemble of turbulent helical swirls should have a net effect of this kind
  - Cowling’s theorem does not apply as each swirl is localized (“non-axisymmetric”)

Numerical evidence

- Small-scale helical turbulence can generate large-scale field
  - Critical Rm is O(1), lower than that of the small-scale dynamo

[Brandenburg, ApJ 2001]

[Meneguzzi et al., PRL 1981 — again !]

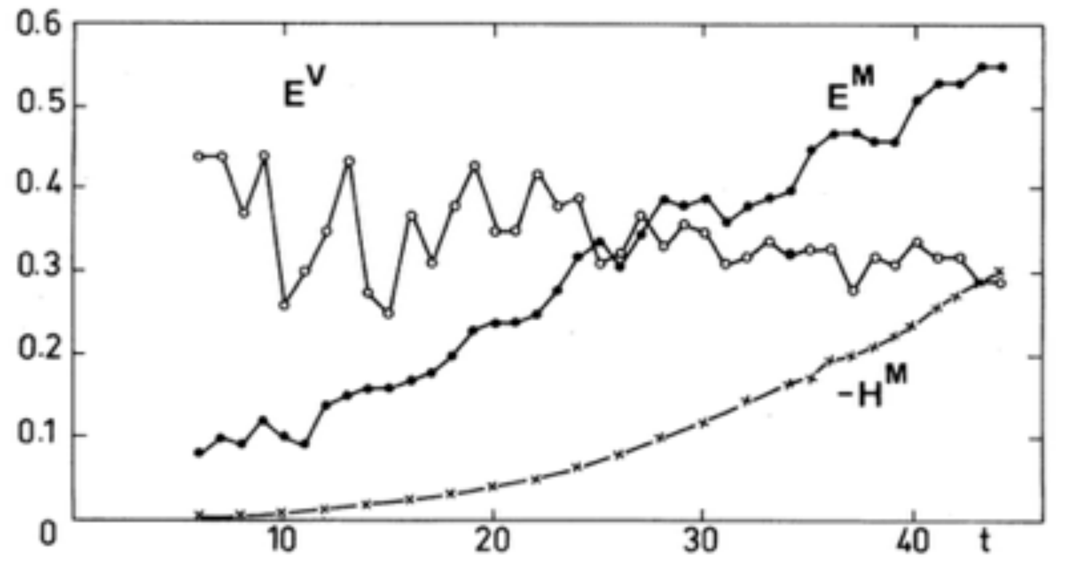
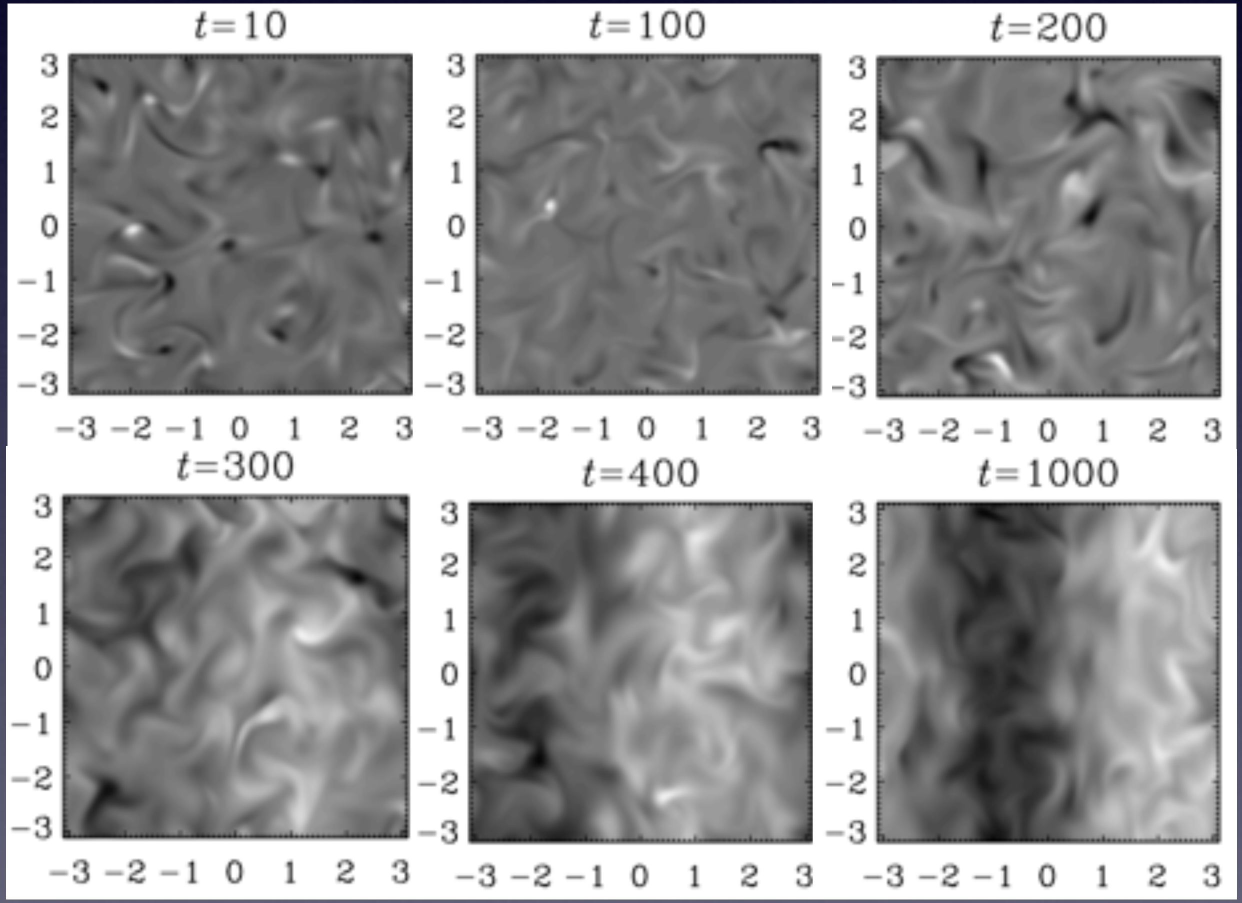


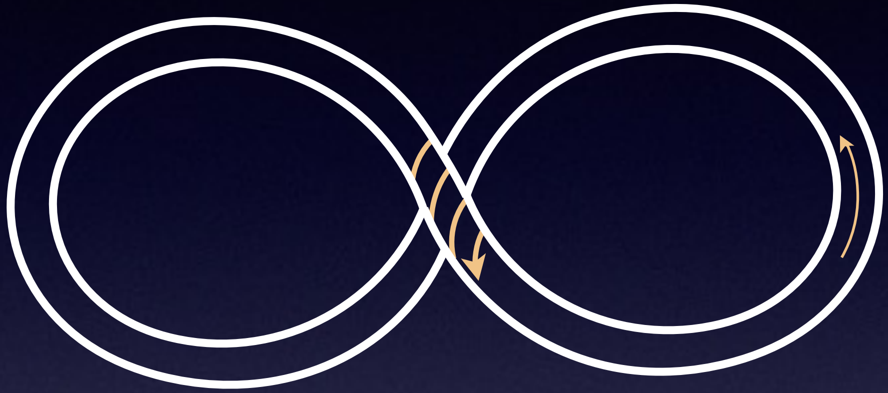
FIG. 4. Helical dynamo with driving at intermediate scales ($k = 5$). Temporal variation of kinetic energy (E^V), magnetic energy (E^M), and magnetic helicity ($-H^M$).

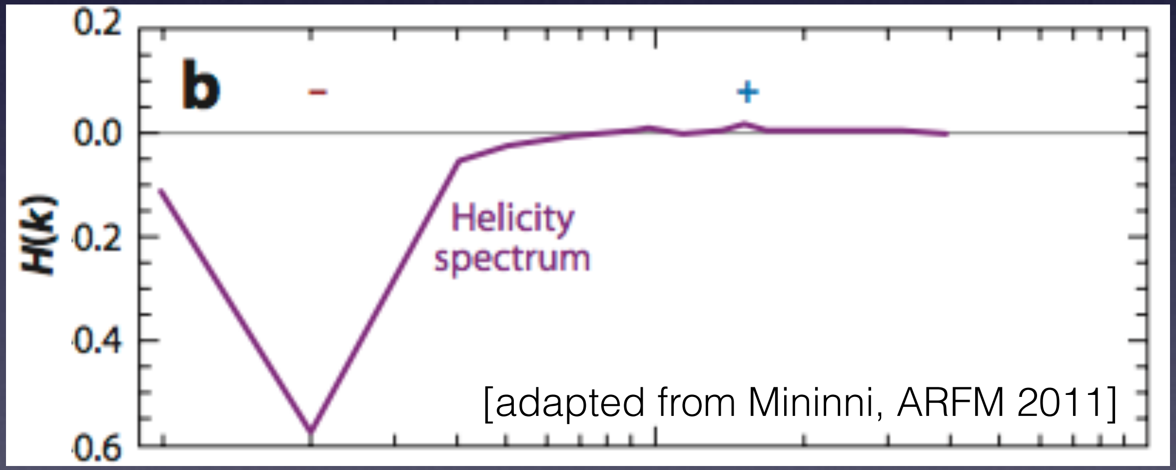


- Helicity seemingly key for large-scale dynamos (but see later)

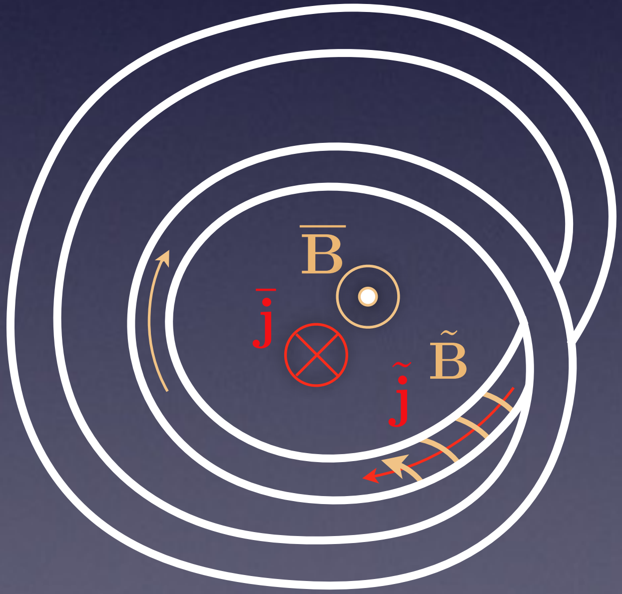
Twisting and magnetic helicity

- Assume conservation of magnetic helicity (up to resistive effects)
- Systematic twisting produces
  - negative large-scale magnetic helicity
  - positive small-scale magnetic helicity





[adapted from Mininni, ARFM 2011]



- Consequences
  - Interpretation of large-scale helical dynamo as “inverse transfer” of helicity
  - Transfer of helicity at small scales

[Frisch et al., JFM 1975]

Mean-field approach

- Incompressible, kinematic problem with uniform diffusivity

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\Delta\mathbf{B}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \nabla\cdot\mathbf{B} = 0$$

- Split fields into large-scale $(\ell > \ell_0)$ and fluctuating part $(\ell < \ell_0)$

$$\mathbf{B} = \overline{\mathbf{B}} + \tilde{\mathbf{B}} \qquad \mathbf{u} = \overline{\mathbf{u}} + \tilde{\mathbf{u}}$$

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} + \overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{B}} = \overline{\mathbf{B}}\cdot\nabla\overline{\mathbf{u}} + \nabla\times\left(\overline{\tilde{\mathbf{u}}\times\tilde{\mathbf{B}}}\right) + \eta\Delta\overline{\mathbf{B}}$$

- To determine the evolution of $\overline{\mathbf{B}}$ we need to know $\overline{\mathcal{E}} = \overline{\tilde{\mathbf{u}}\times\tilde{\mathbf{B}}}$
  - We cannot just sweep fluctuations under the rug: closure problem

[Any good review covers this, see references]

Mean-field approach

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \nabla \times \left[\left(\tilde{\mathbf{u}} \times \overline{\mathbf{B}} \right) + \left(\overline{\mathbf{u}} \times \tilde{\mathbf{B}} \right) + \underline{\left(\tilde{\mathbf{u}} \times \tilde{\mathbf{B}} \right) - \left(\overline{\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}} \right)} \right] + \eta \Delta \tilde{\mathbf{B}}$$

Tangling/shearing
of mean field

Tricky bit — closure problem !
[also known as the "pain in the neck" term]

- Assume linear relation between $\tilde{\mathbf{B}}$ and $\overline{\mathbf{B}}$
  - Expand $(\overline{\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}})_i = \alpha_{ij}\overline{\mathbf{B}}_j + \beta_{ijk}\nabla_k\overline{\mathbf{B}}_j + \cdots$
  - Simplest pseudo-isotropic case: $\alpha_{ij} = \alpha\delta_{ij}, \;\; \beta_{ijk} = \beta\epsilon_{ijk}$

[Warning: hard to justify if
there is small-scale dynamo !]

- For $\overline{u} = 0$, we obtain a closed "α^2" dynamo equation $(\eta \ll \beta)$

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \left(\alpha \overline{\mathbf{B}} \right) + \beta \Delta \overline{\mathbf{B}}$$

alpha effect   beta effect ("turbulent" diffusion)

  - Exponentially growing solutions with real eigenvalues $\gamma = |\alpha| k - \beta k^2$
  - Max growth rate $\gamma_{\max} = \alpha^2/(4\beta)$ at scale $\ell_{\max} = 2\beta/\alpha \gg \ell_0$

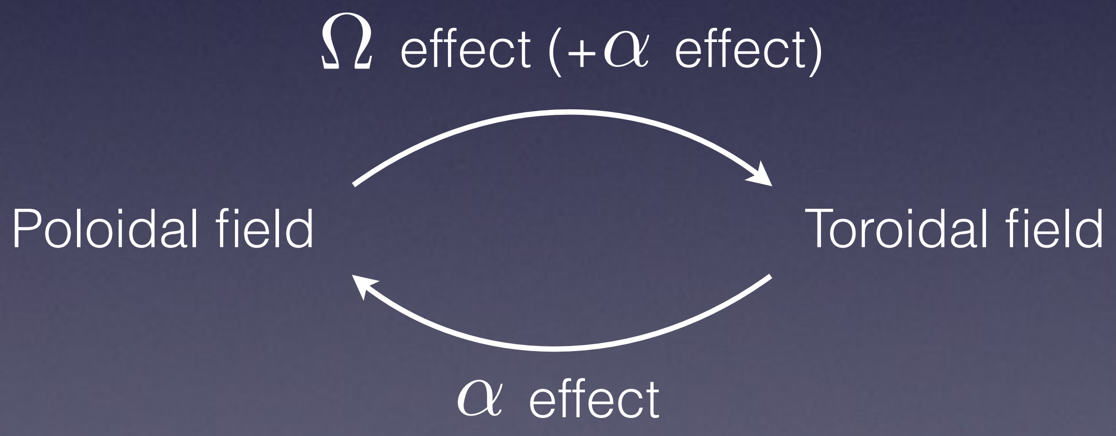
Mean-field dynamo with Omega effect

- Add large-scale differential rotation to MF equation: $\overline{\mathbf{u}} = r\Omega(r,z)\mathbf{e}_\varphi$

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \mathbf{e}_\varphi r \left(\mathbf{B}_{\rm pol} \cdot \nabla\right) \Omega + \nabla \times \left(\alpha \overline{\mathbf{B}}\right) + \beta \Delta \overline{\mathbf{B}}$$

Omega effect Alpha effect

  - Growing, oscillatory solutions leading to field reversals: Parker waves
  - This is called the $\alpha\Omega$ dynamo ($\alpha^2\Omega$ if α acts both ways)



Ω effect (+α effect)

Poloidal field

Toroidal field

α effect

- Remarks
  - Many other couplings possible: pumping effects, non-diagonal terms etc.
  - 3Dness of the dynamo is hidden in mean-field coefficients

Calculation of mean-field coefficients

- We only know how to calculate α and β perturbatively for
  - small correlation times (low Strouhal number $\tau_c/\tau_{\rm NL}$, random waves)
  - low magnetic Reynolds number $\mathrm{Rm} \sim \tau_\eta/\tau_{\rm NL} \ll 1$

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \nabla \times \left[\left(\tilde{\mathbf{u}} \times \overline{\mathbf{B}}\right) + \left(\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}\right) - \left(\overline{\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}}\right) \right] + \eta \Delta \tilde{\mathbf{B}} \qquad (\overline{\mathbf{u}} \doteq 0)$$

$$O(\tilde{B}_{\rm rms}/\tau_c) \qquad O(\overline{B}/\tau_{\rm NL}) \qquad \overline{\quad O(\tilde{B}_{\rm rms}/\tau_{\rm NL}) \quad} \qquad O(\tilde{B}_{\rm rms}/\tau_\eta)$$

$\tau_{\rm NL} = \ell_u/u_{\rm rms}$ tricky "pain in the neck" term G $\tau_\eta = \ell_u^2/\eta$

- In both cases we can justify neglecting the tricky term
  - First Order Smoothing Approximation (FOSA, SOCA, Born, quasilinear…)

[Steenbeck et al., Astr. Nach. 1966; see H. K. Moffatt's textbook, CUP 1978;
Brandenburg & Subramanian, Phys. Rep. 2005]

Calculation of mean-field coefficients

- Let's see how the calculation for $\tau_c/\tau_{\rm NL} \ll 1$
  - Neglecting the tricky term and assuming small resistivity,

$$\begin{aligned}\overline{\tilde{\mathbf{u}}(t)\times\tilde{\mathbf{B}}(t)} &= \overline{\tilde{\mathbf{u}}(t)\times\int_0^t \nabla\times\left[\tilde{\mathbf{u}}(t')\times\overline{\mathbf{B}}(t')\right]dt'}\\ &= \int_0^t\left[\hat{\alpha}(t-t')\overline{\mathbf{B}(t')}-\hat{\beta}(t-t')\nabla\times\overline{\mathbf{B}}\right]dt' \qquad \text{(isotropic case)}\end{aligned}$$

$$\hat{\alpha} = \frac{1}{3}\overline{\tilde{\mathbf{u}}(t)\cdot\tilde{\boldsymbol{\omega}}(t')} \qquad \hat{\beta} = \frac{1}{3}\overline{\tilde{\mathbf{u}}(t)\cdot\tilde{\mathbf{u}}(t')} \qquad \tilde{\boldsymbol{\omega}} = \nabla\times\tilde{\mathbf{u}}$$

  - For slowly varying $\overline{\mathbf{B}}$ and short-correlated velocities, this simplifies as

$$\overline{\tilde{\mathbf{u}}(t)\times\tilde{\mathbf{B}}(t)} = \alpha\overline{\mathbf{B}} - \beta\nabla\times\overline{\mathbf{B}}$$

$$\alpha \simeq -\frac{1}{3}\tau_c\overline{(\tilde{\mathbf{u}}\cdot\tilde{\boldsymbol{\omega}})} \qquad \beta \simeq \frac{1}{3}\tau_c\overline{\tilde{\mathbf{u}}^2}$$

  - The role of kinetic helicity is explicit
- At low Rm, we have the similar result $\alpha \simeq -\frac{1}{3}\tau_\eta\overline{(\tilde{\mathbf{u}}\cdot\tilde{\boldsymbol{\omega}})}$

Dynamical regime of large-scale dynamos

- When B gets “large enough”, the Lorentz force back-reacts
  - Big questions: what happens then, and what is “large-enough” ?

    [Brandenburg & Subramanian, Phys. Rep. 2005, and refs. therein: Proctor, 2003; Diamond et al. 2005]
- Equipartition argument: saturation when $\overline{\mathbf{B}}^2 \sim 4\pi\overline{\rho\,\tilde{\mathbf{u}}^2} \equiv B_{\rm eq}^2$, but
  - $\overline{\mathbf{B}}$ and $\tilde{\mathbf{u}}$ have very different scales
  - Large-scale dynamos alone produce plenty of small-scale field
- Equipartition of small-scale fields: $\overline{\tilde{\mathbf{b}}^2} \sim B_{\rm eq}^2$, with $\overline{\tilde{b}^2} \sim \mathrm{Rm}^p\overline{B}^2$
  - Not very astro-friendly: $\overline{\mathbf{B}}^2 \sim B_{\rm eq}^2/\mathrm{Rm}^p \ll B_{\rm eq}^2$ for *p=O(1)*
  - Possibility of “catastrophic” alpha quenching

$$\alpha(\overline{\mathbf{B}}) = \frac{\alpha_0}{1+\mathrm{Rm}^q\left(\overline{\mathbf{B}}^2/B_{\rm eq}^2\right)} \qquad q = O(1)$$

The quenching issue

- Physical origin of quenching “vigorously” debated:
  - Magnetized fluid has “memory”: possible drastic reduction of statistical effects compared to random walk estimates [see review by Diamond et al., 2005]
  - Magnetic helicity conservation argument:
    - in “closed” systems, large-scale field can only reach equipartition on slow, large-scale resistive timescales [e.g. Brandenburg, ApJ 2001]
- Possible way out of problem is to evacuate magnetic helicity [Blackman & Field, ApJ 2000; see discussion by Brandenburg, Space Sci. Rev. (2009)]

$$\frac{d}{dt}\langle \mathbf{A}\cdot\mathbf{B}\rangle_V = -2\eta\,\langle\, (\nabla\times\mathbf{B})\cdot\mathbf{B}\rangle_V - \langle \nabla\cdot\mathbf{F}_{\mathcal{H}_m}\rangle$$

  - Requires open boundary conditions (periodic b.c. not ok)
  - Requires internal fluxes of helicity [Kleeorin et al., Vishniac-Cho etc.]

Transitional (yet important) remarks

- Historically, mean-field models have been at the core of modelling of
  - solar and stellar dynamos — "alpha" provided by cyclonic convection
  - galactic dynamos — "alpha" provided by supernova explosions

- But classical mean-field theory faces strong limitations
  - Astro turbulence typically has $\tau_c/\tau_{\mathrm{NL}} \sim 1$ and $\mathrm{Rm} \gg 1$
  - "Co-existence" with fast, small-scale dynamo for $\mathrm{Rm} \gg 1$
    - pain in the neck term exponentially growing…then what ?
    - linear relation between $\tilde{\mathbf{b}}$ and $\overline{\mathbf{B}}$ doubtful

- Large-scale dynamos are "real" — independently of our limited theories
  - We have to think harder ! (and ask good questions to computers)

Connecting both worlds

Large-scale dynamos with Kasantsev

[Vainshtein & Kitchatinov, JFM 1986,
Berger & Rosner, GAFD 1995,
Subramanian, PRL 1999,
Boldyrev et al., PRL 2005]

- Consider turbulence with net helicity
  - Add a mirror symmetry-breaking term to the correlators

$$\kappa^{ij}(\mathbf{r}) = \kappa_N(r)\left(\delta^{ij} - \frac{r^i r^j}{r^2}\right) + \kappa_L(r)\frac{r^i r^j}{r^2} + g(r)\varepsilon^{ijk}r^k$$

$$H^{ij}(\mathbf{x}-\mathbf{x}',t) = H_N(r,t)\left(\delta^{ij} - \frac{r^i r^j}{r^2}\right) + H_L(r,t)\frac{r^i r^j}{r^2} + K(r)\varepsilon^{ijk}r^k$$

- $r \to \infty$ asymptotics of model gives mean-field α^2 equation

$$\frac{\partial\overline{\mathbf{B}}}{\partial t} = \nabla\times\left(\alpha\overline{\mathbf{B}}\right) + (\eta+\beta)\Delta\overline{\mathbf{B}} \qquad g(0)=\alpha, \quad \beta = \frac{\kappa_L(0)}{2}$$

- Full calculation leads to coupled equations for H_L and K

$$\frac{\partial H_L}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\kappa\frac{\partial H_L}{\partial r}\right) + GH_L - 4hK \qquad h(r) = g(0) - g(r)$$

$$\frac{\partial K}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\frac{\partial}{\partial r}\left(\kappa K + hH_L\right)\right] \qquad G(r) = \kappa'' + 4\kappa'/r$$

Self-adjoint spinorial form

$$\frac{\partial \mathbf{W}}{\partial t} = -\tilde{\mathsf{R}}^T \tilde{J}\tilde{\mathsf{R}}\mathbf{W}$$

$$A(r) = \sqrt{2}\,[2\eta + \kappa_N(0) - \kappa_N(r)]$$
$$B(r) = \sqrt{2}\,[2\eta + \kappa_L(0) - \kappa_L(r)]$$
$$C(r) = \sqrt{2}\,[g(0) - g(r)]\,r$$

$$\tilde{\mathsf{R}} = \begin{pmatrix} \sqrt{2}/r & 0 \\ 0 & -\dfrac{1}{r^2}\dfrac{\partial}{\partial r}r^2 \end{pmatrix} \qquad \tilde{\mathsf{J}} = \begin{pmatrix} \hat{E} & C \\ C & B \end{pmatrix} \qquad \hat{E} = -\frac{1}{2}r\frac{\partial}{\partial r}B\frac{\partial}{\partial r}r + \frac{1}{\sqrt{2}}(A - rA')$$

$$\frac{\partial}{\partial t}\begin{pmatrix} W_H \\ W_K \end{pmatrix} = \begin{bmatrix} -\dfrac{\sqrt{2}}{r}\hat{E}\dfrac{\sqrt{2}}{r} & \dfrac{\sqrt{2}}{r^3}C(r)\dfrac{\partial}{\partial r}r^2 \\ -r^2\dfrac{\partial}{\partial r}C(r)\dfrac{\sqrt{2}}{r^3} & r^2\dfrac{\partial}{\partial r}\dfrac{B(r)}{r^4}\dfrac{\partial}{\partial r}r^2 \end{bmatrix}\begin{pmatrix} W_H \\ W_K \end{pmatrix}$$

$$H_L = \frac{\sqrt{2}}{r^2}W_H$$
$$K = -\frac{\sqrt{2}}{r^4}\frac{\partial}{\partial r}(r^2 W_K)$$

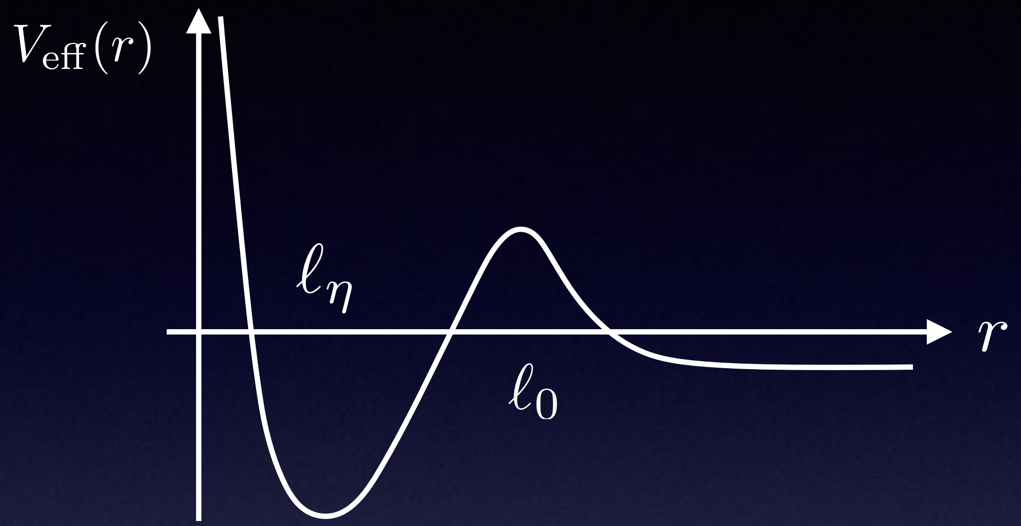
- Therefore, the generalized helical case can be diagonalized
  - Bound “small-scale” modes: $\gamma_n > \gamma_0$
  - Free “mean-field” modes: $0 < \gamma < \gamma_0$

$$\gamma_0 = \frac{g^2(0)}{\kappa_L(0) + 2\eta} = \frac{2\alpha^2}{4(\beta + \eta)}$$

Twice the maximum α^2
mean-field dynamo growth rate !

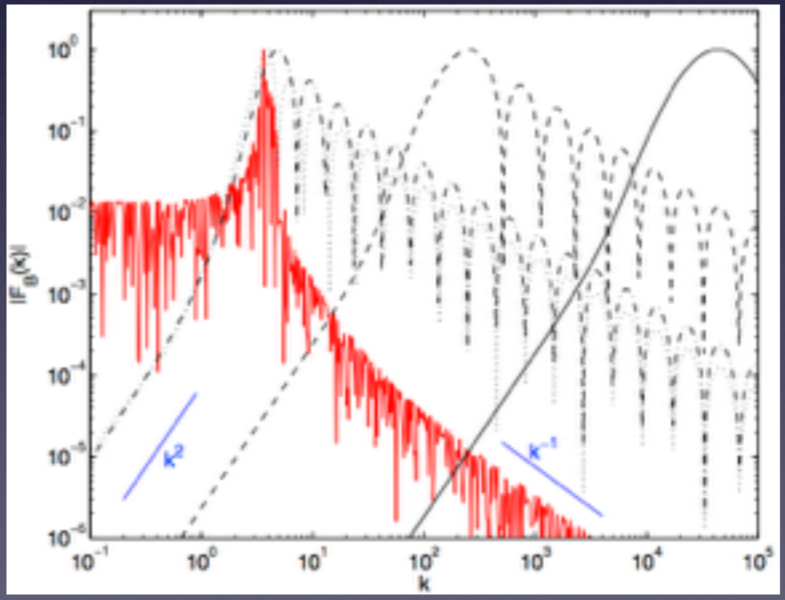
Growing helical modes

- Helicity allows growing large-scale modes
  - $V_{\rm eff}(r) = 2/r^2 - \alpha^2/(\beta+\eta)^2, \quad \ell_0 \ll r$



- Bound modes ($\gamma > \gamma_0$) dominate the kinematic stage
  - As $\gamma \to \gamma_0$, their spectrum peak shifts towards that of “mean-field” modes

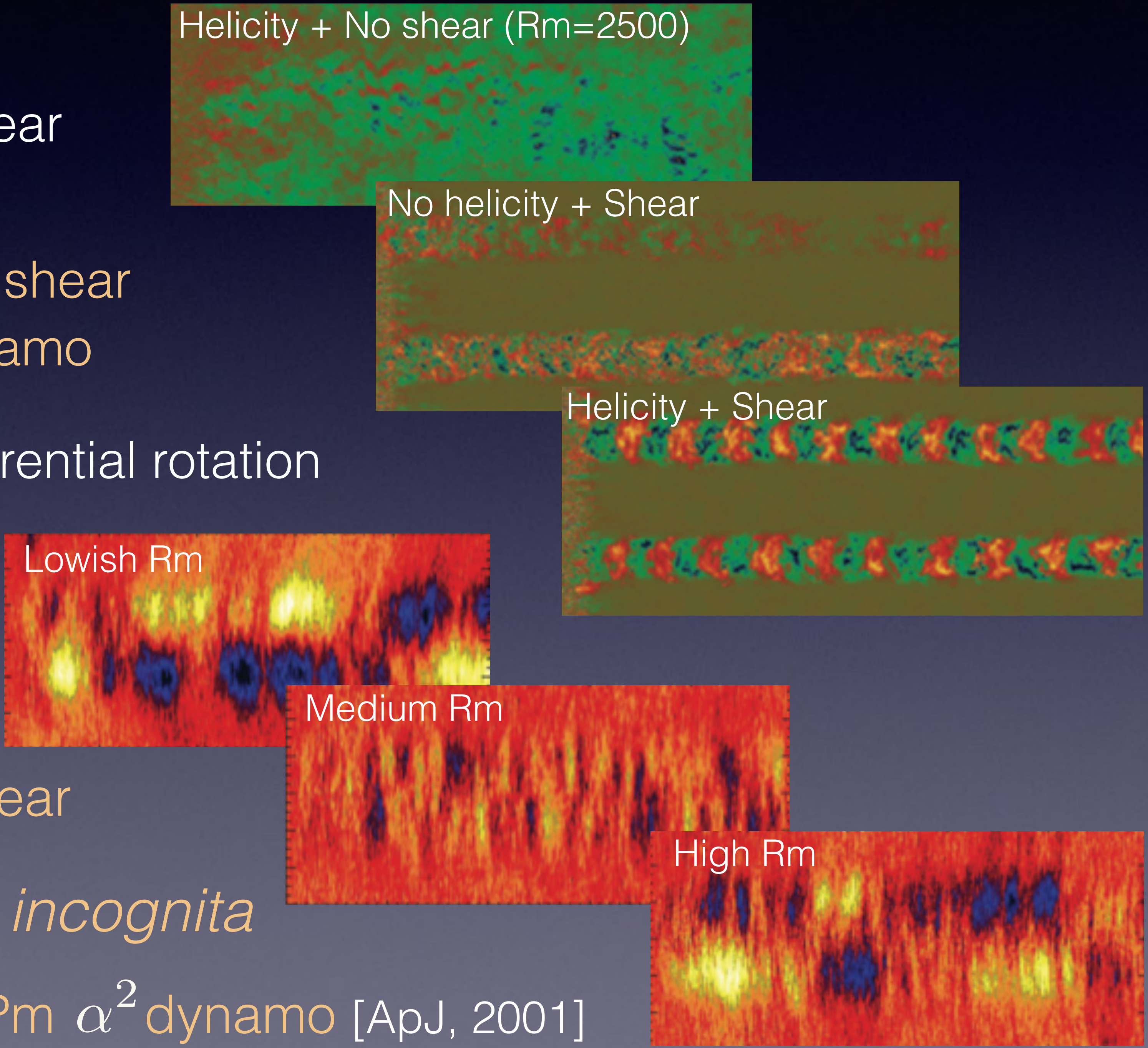
[Malyshkin & Boldyrev, ApJ 2009]



- Further hints that quantitative large-scale dynamo theory should factor in the small-scale dynamo

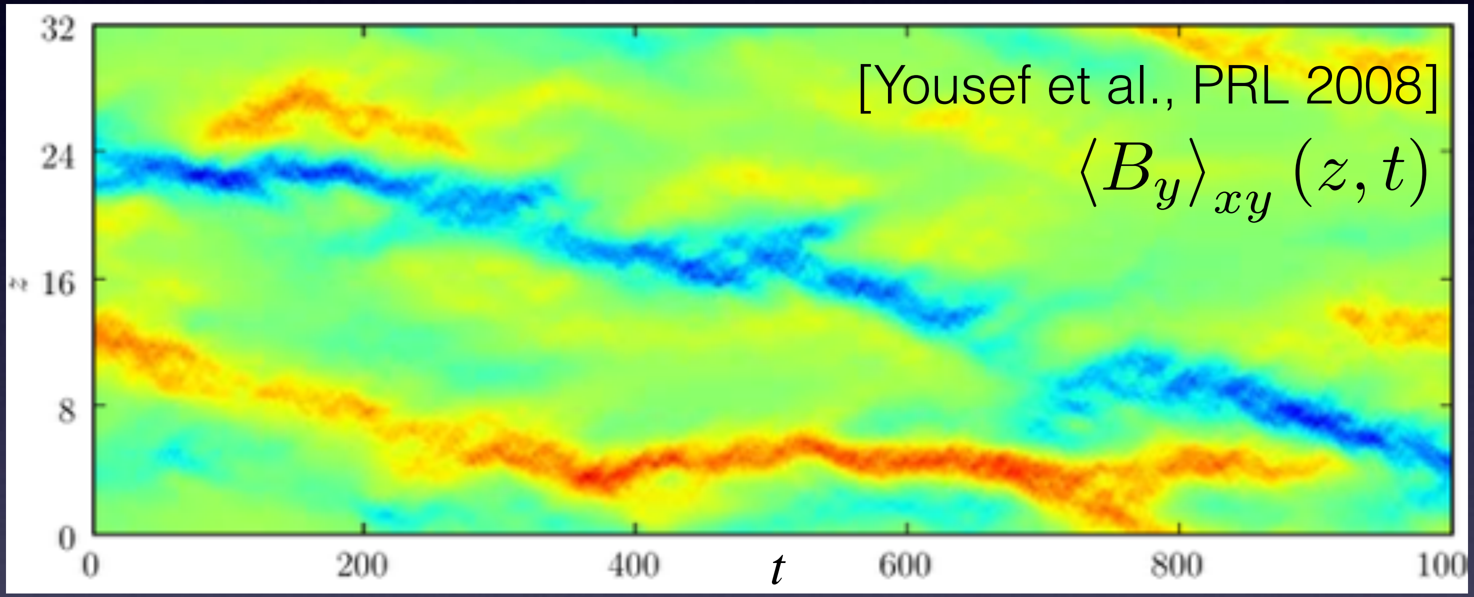
Order out of chaos ?

- Large-scale dynamos at largish Rm now observed numerically
  - Galloway-Proctor flow + Shear [Tobias & Cattaneo, Nature 2013]
    - “Suppression” principle: shear turns off small-scale dynamo
  - Turbulent convection + differential rotation [Hotta et al., Science 2016]
    - Small-scale dynamo reduces turbulence
  - Asymptotic behaviour unclear
- Dynamical theory still *terra incognita*
  - Boldyrev’s model of large Pm α^2 dynamo [ApJ, 2001]



One last (lack of) twist

- Large-scale dynamo action is possible without net helicity
  - The shear dynamo: $\mathbf{u} = Sx\mathbf{e}_y$ + non-helical small-scale turbulence



[Yousef et al., PRL 2008]

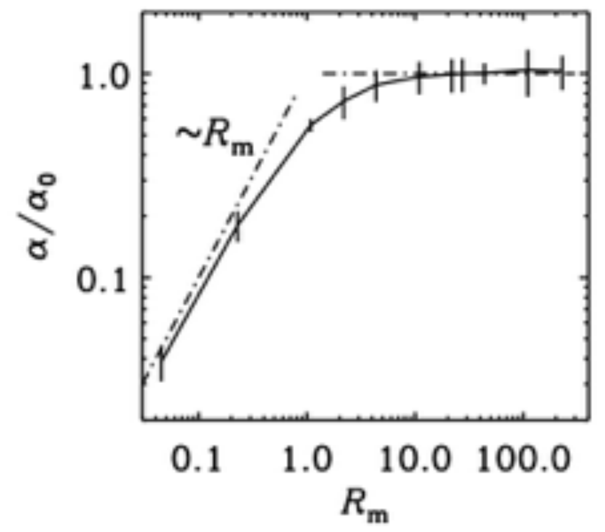
$\langle B_y \rangle_{xy} (z,t)$

  - Mean-field description in terms of “WxJ” effect [Kleeorin & Rogachevskii]
  - “Incoherent” alpha effect [Silant’ev 2007, Proctor 2007, Brandenburg 2008], etc.
- Recent developments [Squire & Battacharjee, PRL 2015]
  - Saturated small-scale dynamo in a shear flow can lead to large-scale dynamo

A few words on “test field”-like methods

- Pragmatic strategies have been devised for “astrophysical applications”
  - postulate generalised mean-field form for $\overline{\boldsymbol{\mathcal{E}}}(\overline{\mathbf{B}})$ (convolution integrals)
  - Measure effective transport coefficients in local simulations
  - Use the results in simpler 2D mean-field models

[Sur et al., MNRAS 2008,
Brandenburg, Space Sci. Rev. 2009]



- Such procedures
  - produce converged values of transport coefficients
  - reproduce exact results in perturbative kinematic limits

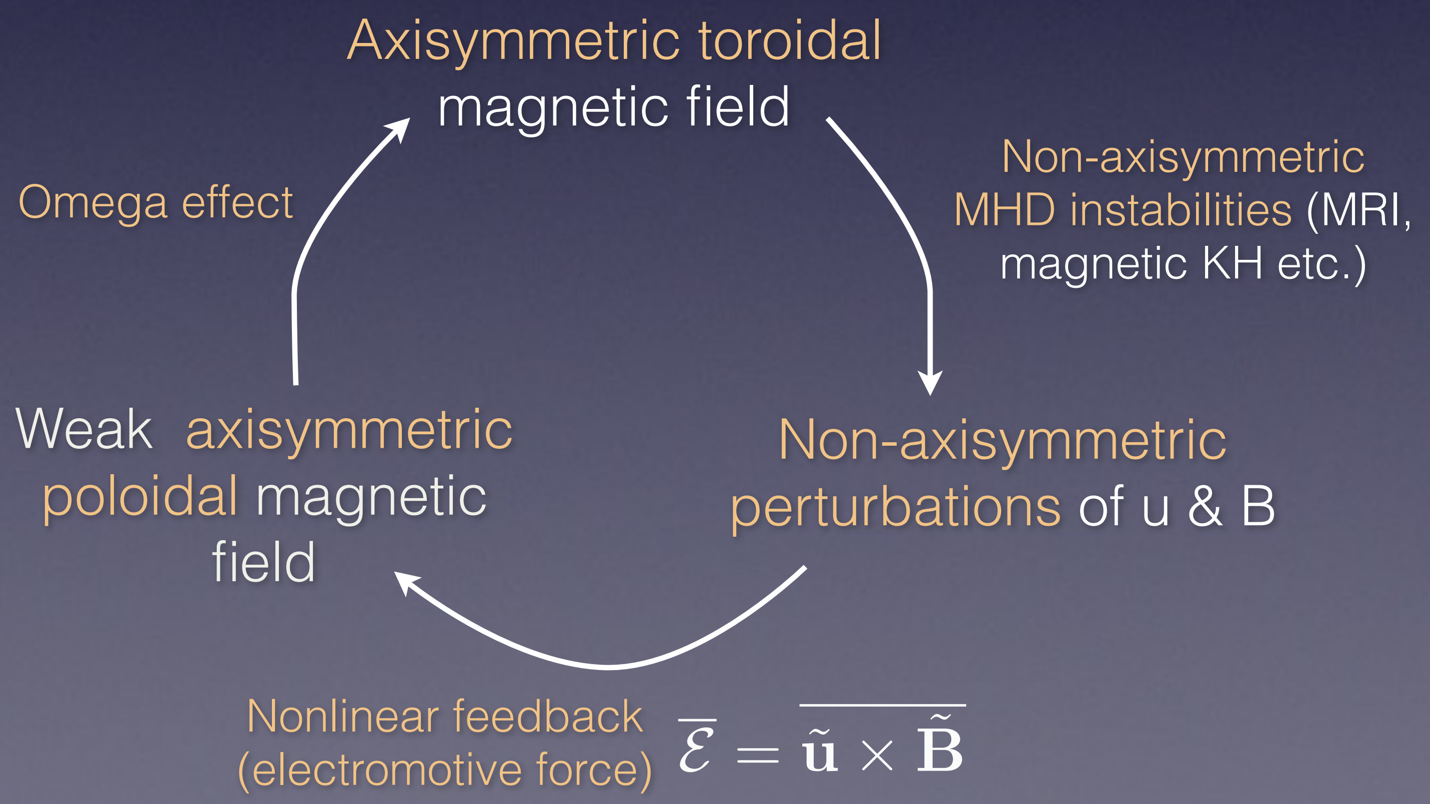
- TFM-based modelling may be useful, but:
  - no rigorous justification as to why it should be accurate/appropriate (Rm>>1 !)
  - dynamical, tensorial convolution relations $\overline{\boldsymbol{\mathcal{E}}}(\overline{\mathbf{B}})$ can fit complex dynamics, but could well be degenerate with more physically-grounded nonlinear models
  - it can obfuscate the underlying physics, e.g. when MHD instabilities are involved

More ways to make magnetic fields:

instability-driven dynamos

Instability-driven dynamos

- Many astrophysical systems
  - host differential rotation: i.e. there is a background shear flow
  - are prone to non-axisymmetric MHD instabilities
- This can lead to specific nonlinear forms of dynamo action
  - Analogous to self-sustaining nonlinear process in hydro shear flows



Axisymmetric toroidal magnetic field

Non-axisymmetric MHD instabilities (MRI, magnetic KH etc.)

Non-axisymmetric perturbations of u & B

Nonlinear feedback (electromotive force) $\overline{\mathcal{E}} = \overline{\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}}$

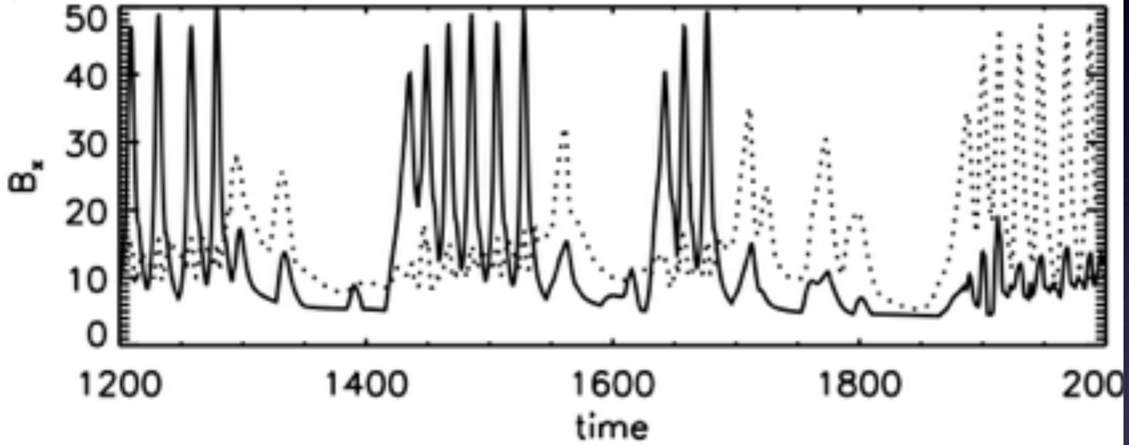
Weak axisymmetric poloidal magnetic field

Omega effect

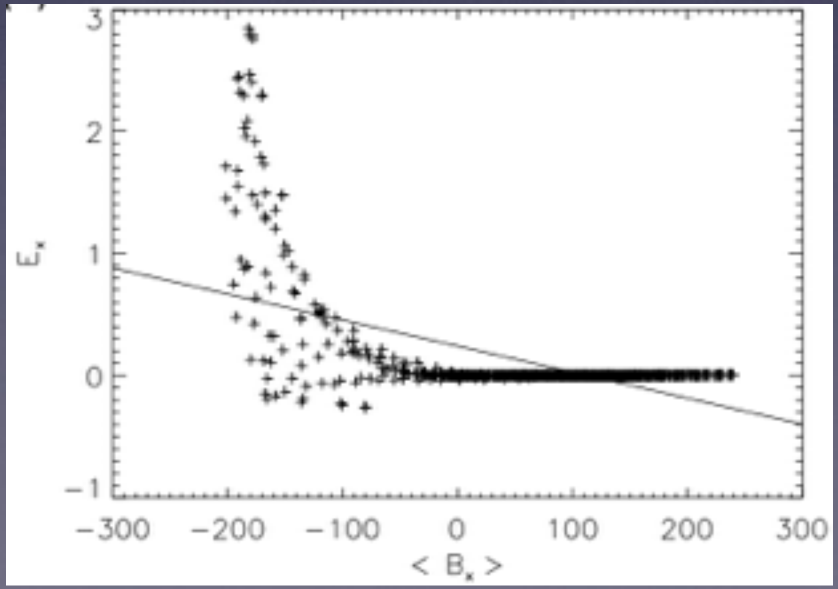
[Rincon et al., PRL 2007;
Astron. Nachr. 2008;
Riols et al., JFM 2013]

“Solar-like” magnetic buoyancy dynamo

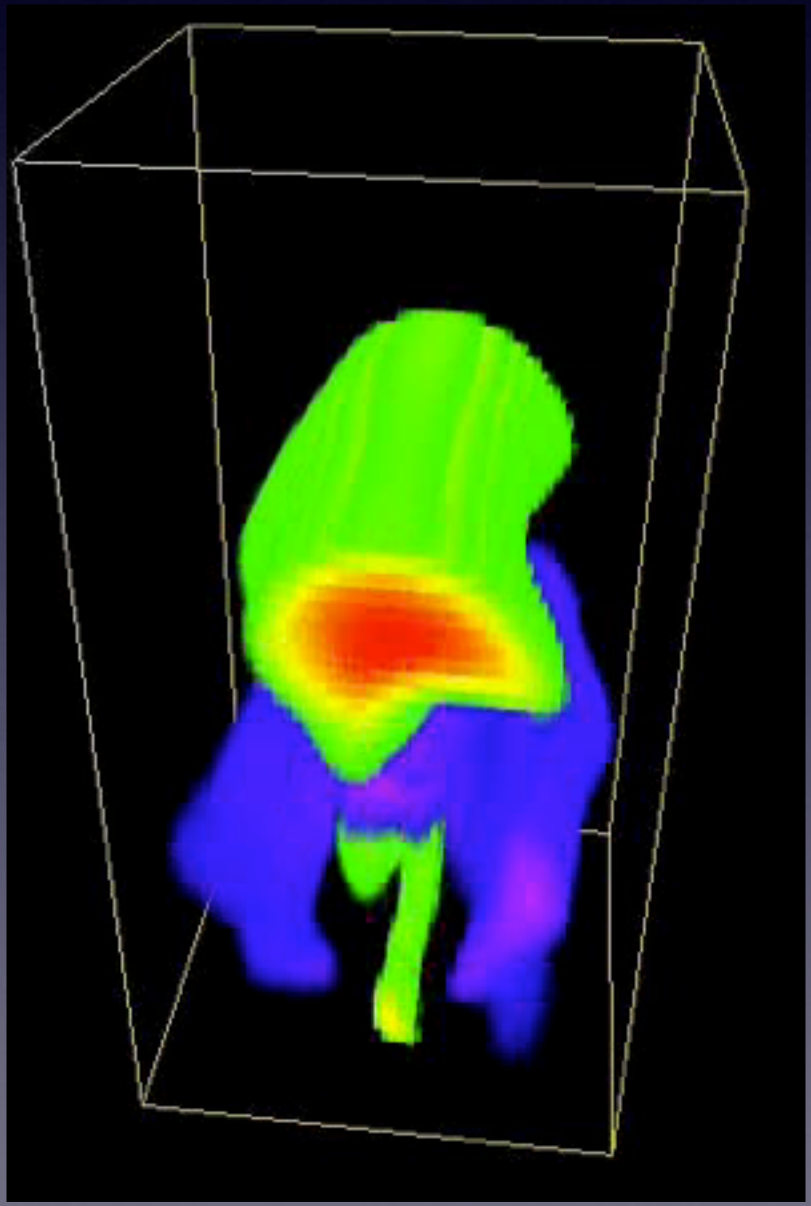
- Shear + Magnetic buoyancy + Kelvin-Helmholtz
  - Coherent, strongly chaotic dynamo action



  - Strongly nonlinear EMF / field relationship

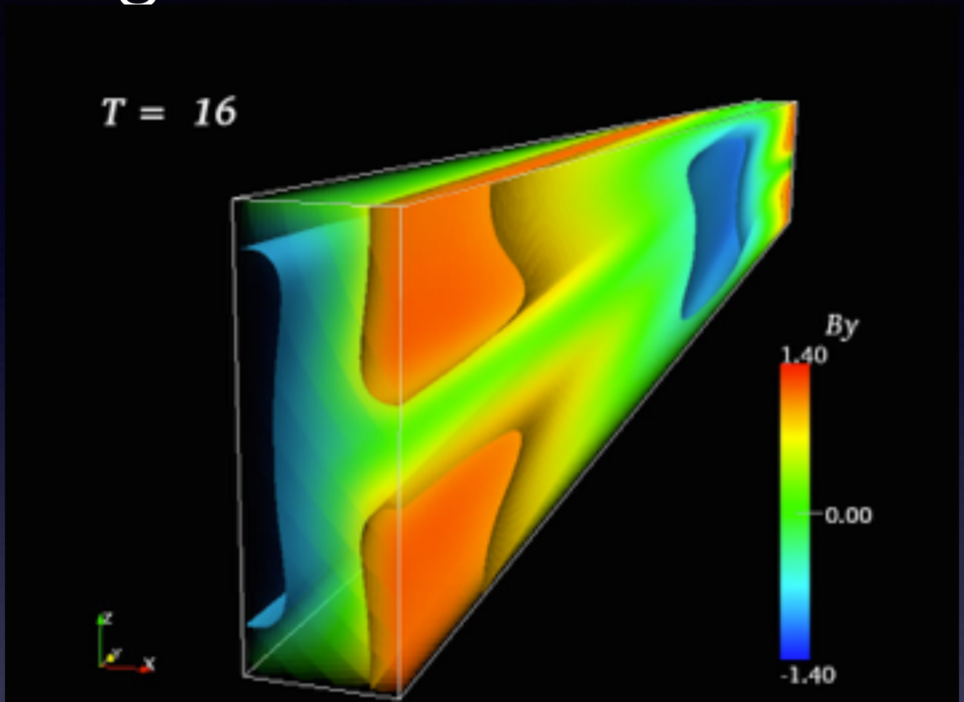


[Cline et al., ApJ 2003]

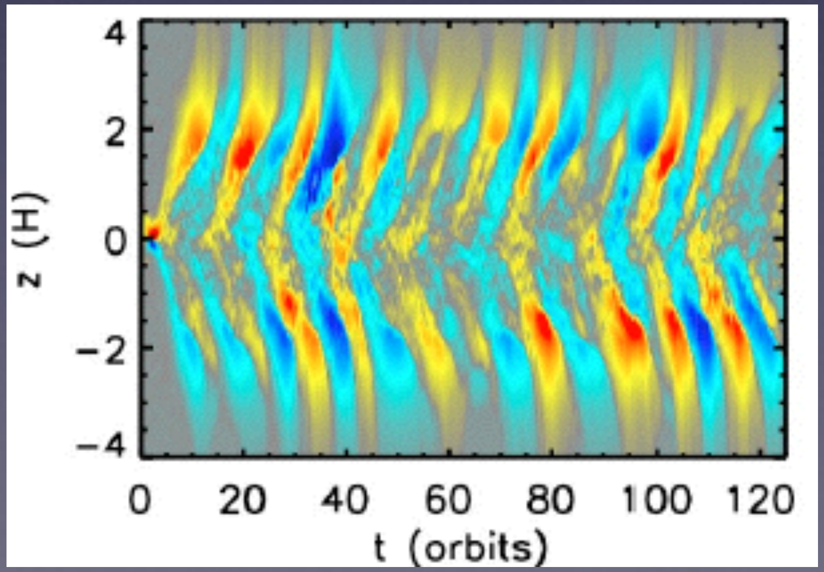


Accretion disk dynamo

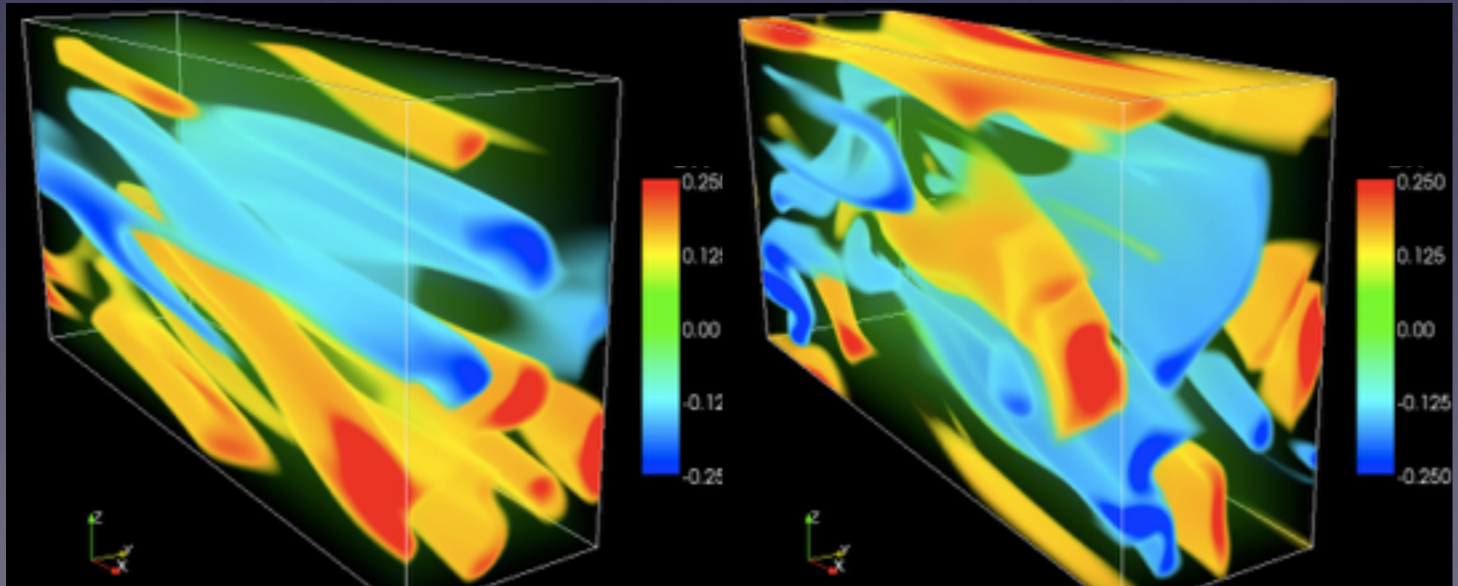
- Keplerian shear flow turbulence is thought to be MRI-driven
    - Possible even in the absence of net magnetic flux [Hawley et al., ApJ 1996]
- Characterised by dynamical reversals of large-scale field
    - Non-axisymmetric MRI of toroidal field critical (magnetic buoyancy)



[Herault et al., PRE 2011]



[Davis et al., ApJ 2010]

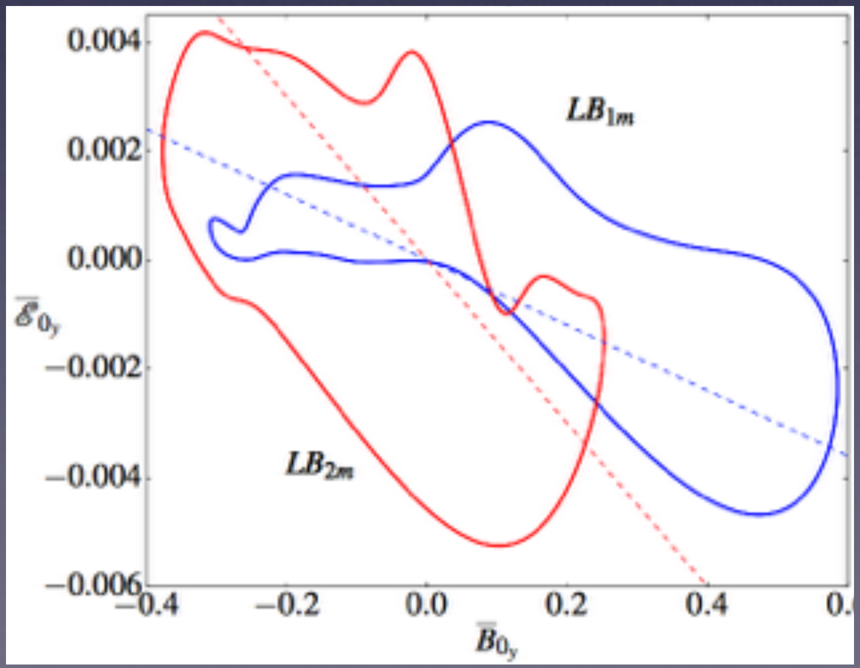


[Lesur & Ogilvie, A&A 2008]

From subcritical to statistical

- Such dynamos are subcritical / essentially nonlinear
  - “Egg and chicken” problem
    - Non-axisymmetric instability growth requires large-scale field
    - Large-scale field sustainement rests on non-axisymmetric instability
  - Non-axisymmetric $\tilde{\mathbf{u}}, \tilde{\mathbf{B}}$ jointly excited by instability: Lorentz force essential

[Riols et al., A&A 2017]

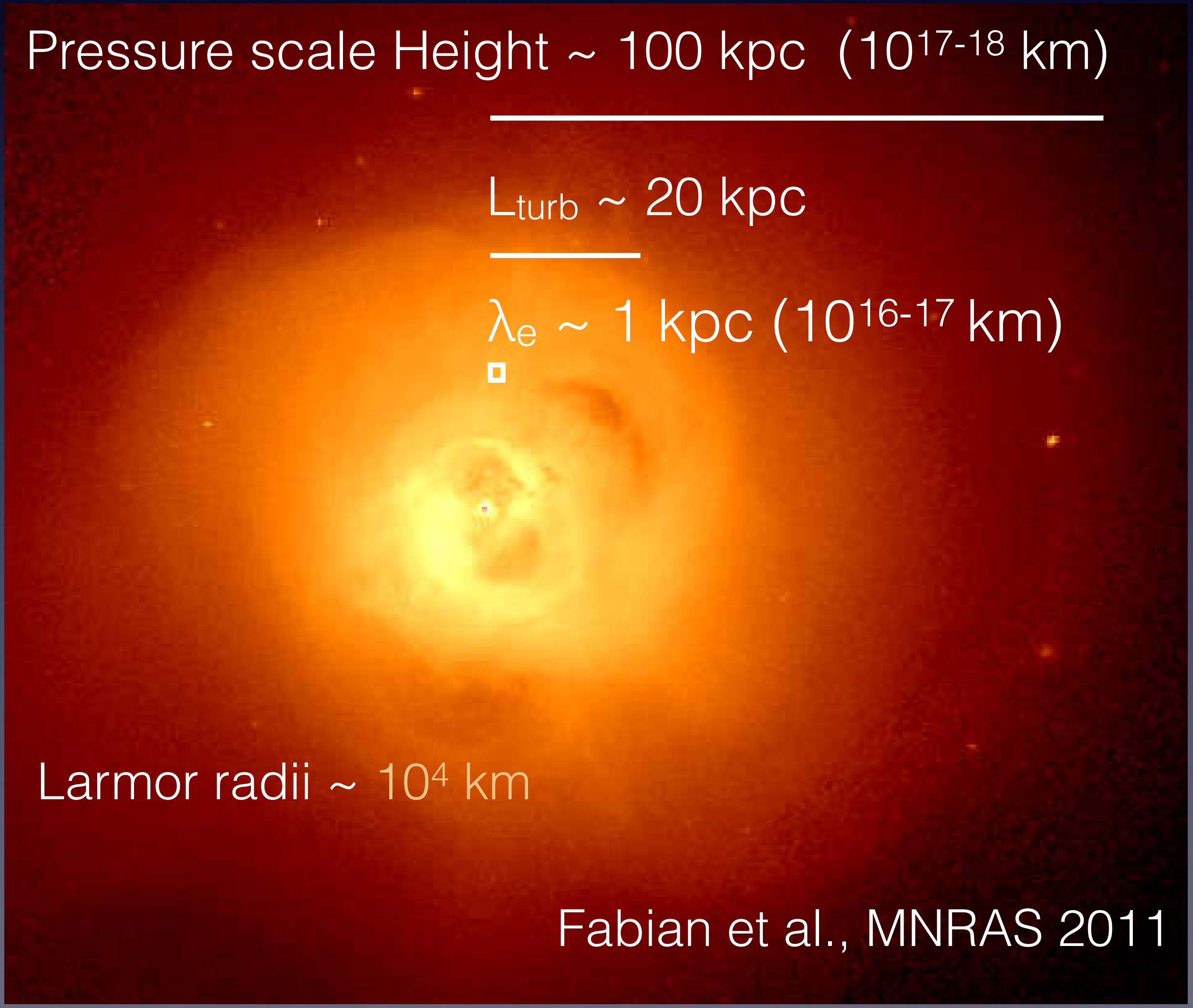


- Implications
  - No kinematic stage, homoclinic bifurcations
  - Nonlinear EMF/field relationship
- Statistical theory relevant but difficult
  - Mean-field approach controversial

Plasma dynamo

What about weakly collisional plasmas ?

- Some high-energy astrophysical plasmas are not MHD fluids
    - Intracluster medium, hot accretion flows, primordial plasma (?)
- What happens to dynamos ?
    - Implications for magnetogenesis
    - “Pathfinding” for experiments
- Coupling of processes
    - Fluid: stirring, fluid instabilities (convection, MRI etc.)
    - Kinetic: collisionless damping, magnetization effects



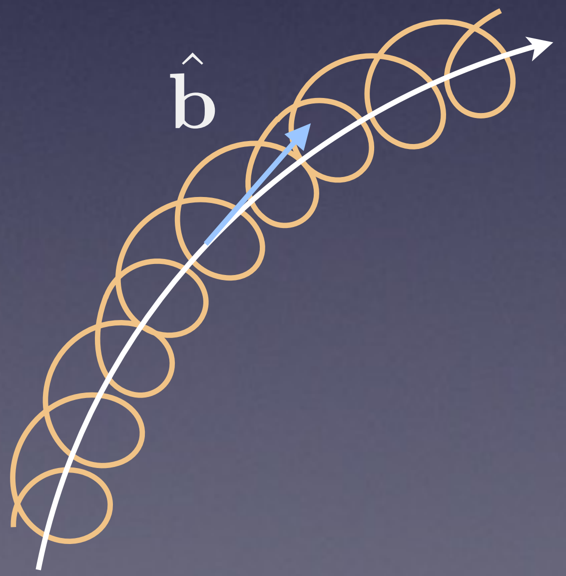
Fabian et al., MNRAS 2011

Pressure anisotropy generation

- In a magnetized, weakly collisional plasma
  - The pressure is an anisotropic tensor with respect to the direction of B
  - $\mu_s = m_s v_\perp^2 / 2B$ is almost conserved
- Large-scale, field-stretching motions generate pressure anisotropy
  - Collisions tend to relax it

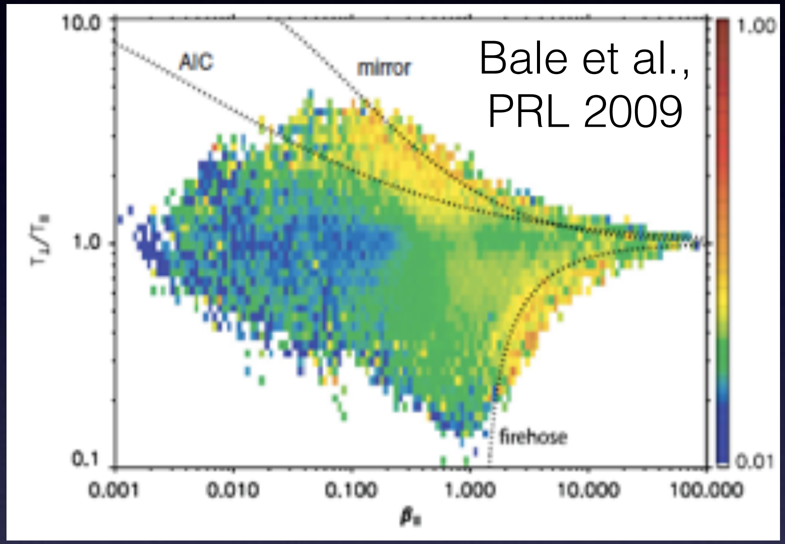
$$\frac{1}{p_\perp}\frac{\mathrm{d}p_\perp}{\mathrm{d}t} \sim \frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} - \nu_{ii}\frac{p_\perp - p_{||}}{p}$$

$$\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} = \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{u}$$

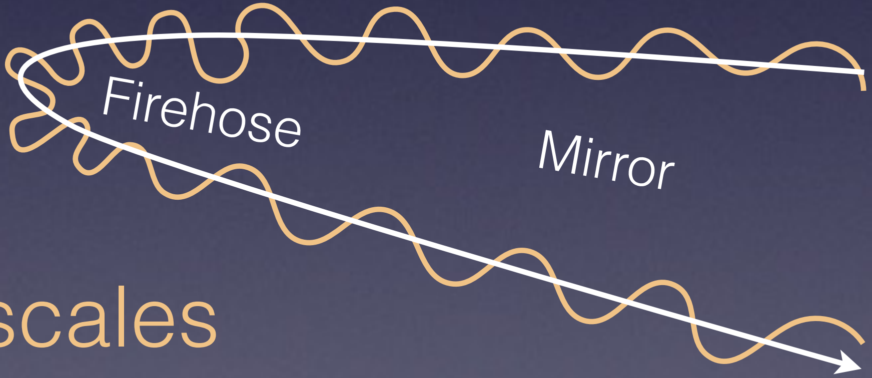


Pressure anisotropy-driven instabilities

- $\mu = mv_\perp^2/2B$ conservation implies kinetic instability everywhere
  - local increase of |B| —> increase of $p_\perp$
    - mirror instable $\dfrac{p_\perp - p_\parallel}{p_\perp} > 1/\beta$
  - local decrease of |B| —> decrease of $p_\perp$
    - firehose instable $\dfrac{p_\perp - p_\parallel}{p_\perp} < -2/\beta$

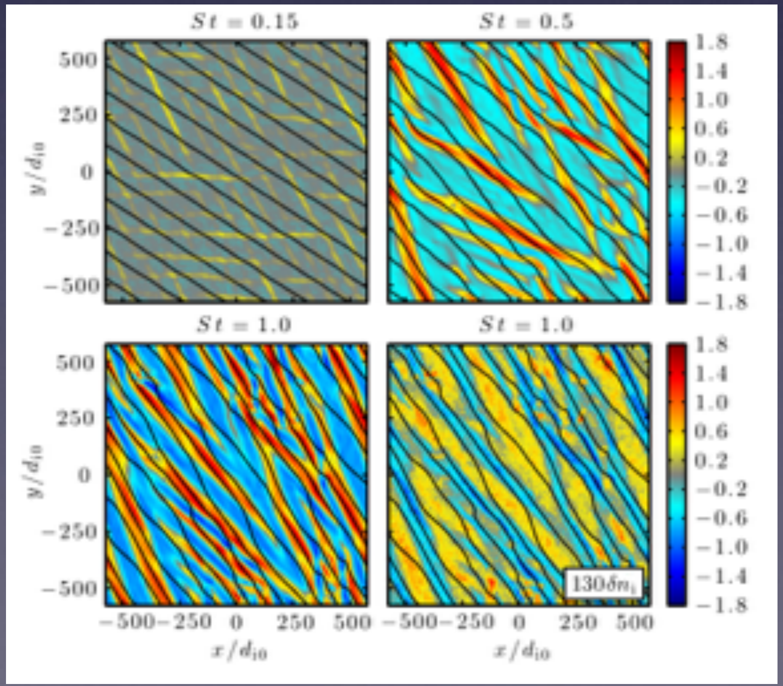


Bale et al., PRL 2009



- Small, fast scales
  - ICM: $\rho_i \sim 10^4$ km, $\Omega_i^{-1} \sim$ second
- Feedback non-linearly on “fluid” scales

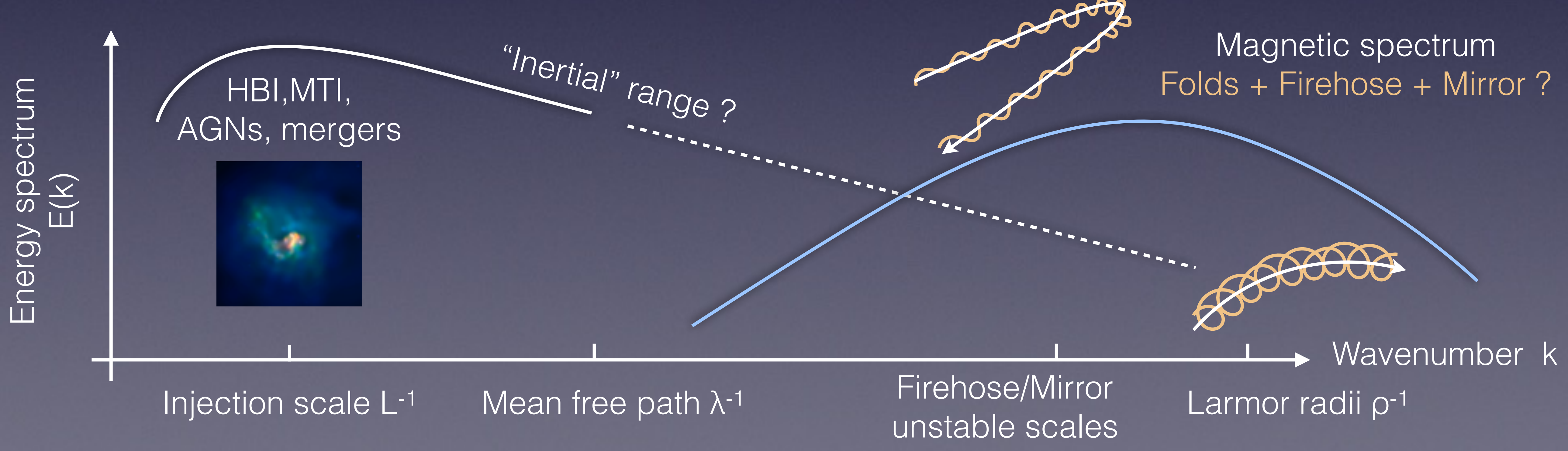
[Scheckochihin et al, ApJ 2005, Schekochihin et al., PRL 2008; Rosin et al., MNRAS 2011; Rincon et al., MNRAS 2015]



[Kunz et al., PRL 2014]

So what happens to dynamos ?

- The most efficient eddies are the smallest, fastest ones
  - In the ICM, such plasma motions are weakly collisional
- Plasma is magnetised well below equipartition (ICM: 10^{-13} G)
  - Field-stretching motions (= dynamo !) generate pressure anisotropy
  - Pressure-anisotropy driven instabilities !

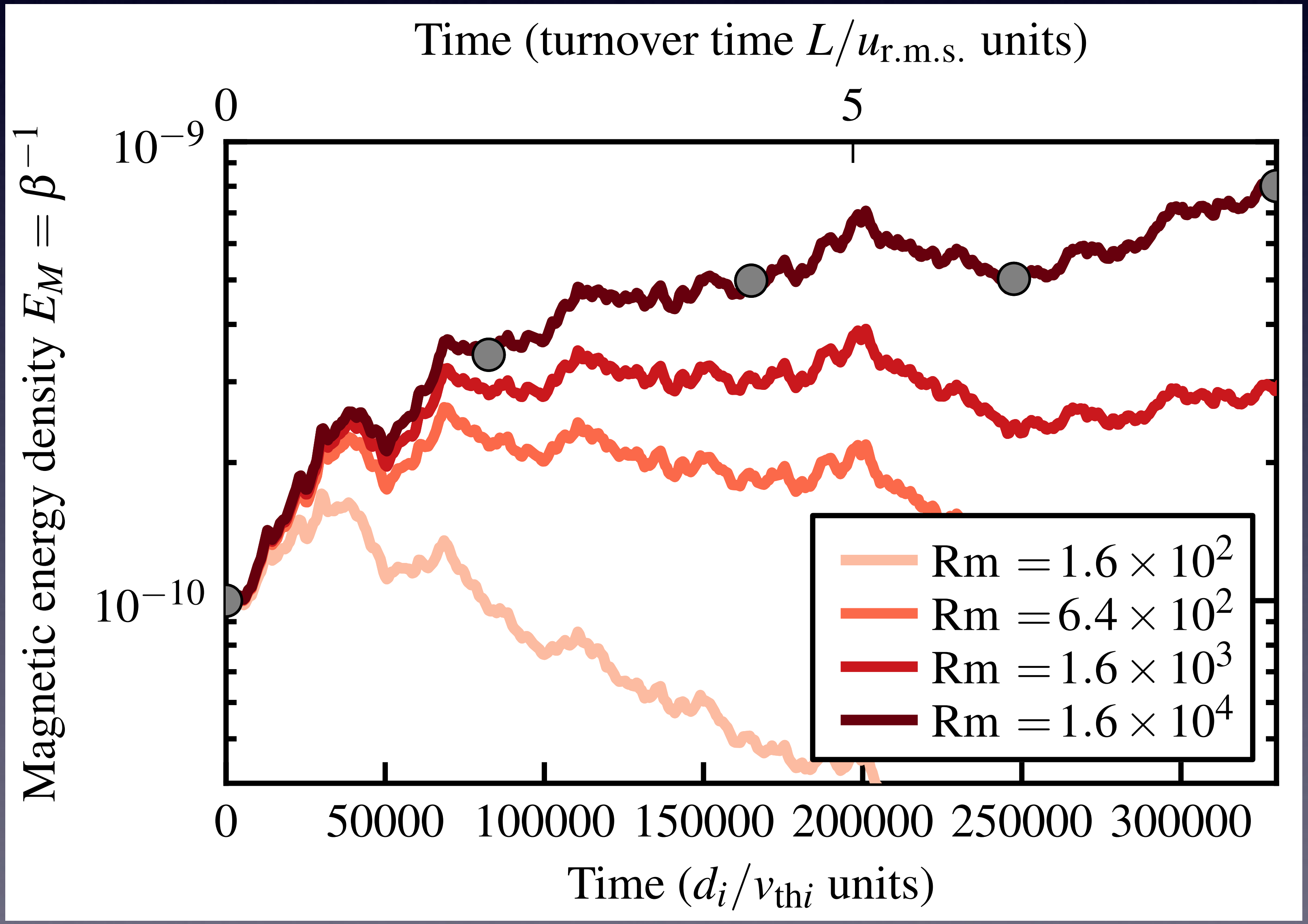


Collisionless plasma dynamo problem(s)

- Unmagnetized problem: $\rho_i/L > 1$
  - Is a collisionless, unmagnetized 3D chaotic flow of plasma a good dynamo ?
- Magnetized problem: $\rho_i/L < 1$
  - How do pressure-anisotropy kinetic instabilities interfere with magnetic growth ?
- Annoying “details”
  - Dynamo is a fundamentally 3D process in physical space (Cowling)
  - No rigid “guide” field here: kinetic description “3V” in velocity space
- Modelling requires 3D-3V simulations (+time integration !)
  - Very costly: O(10^6-10^7 CPU hours) per simulation
  - Use simplest possible appropriate kinetic model

Unmagnetized regime

- Four simulations with same initial field and flow history, but different magnetic diffusivity η



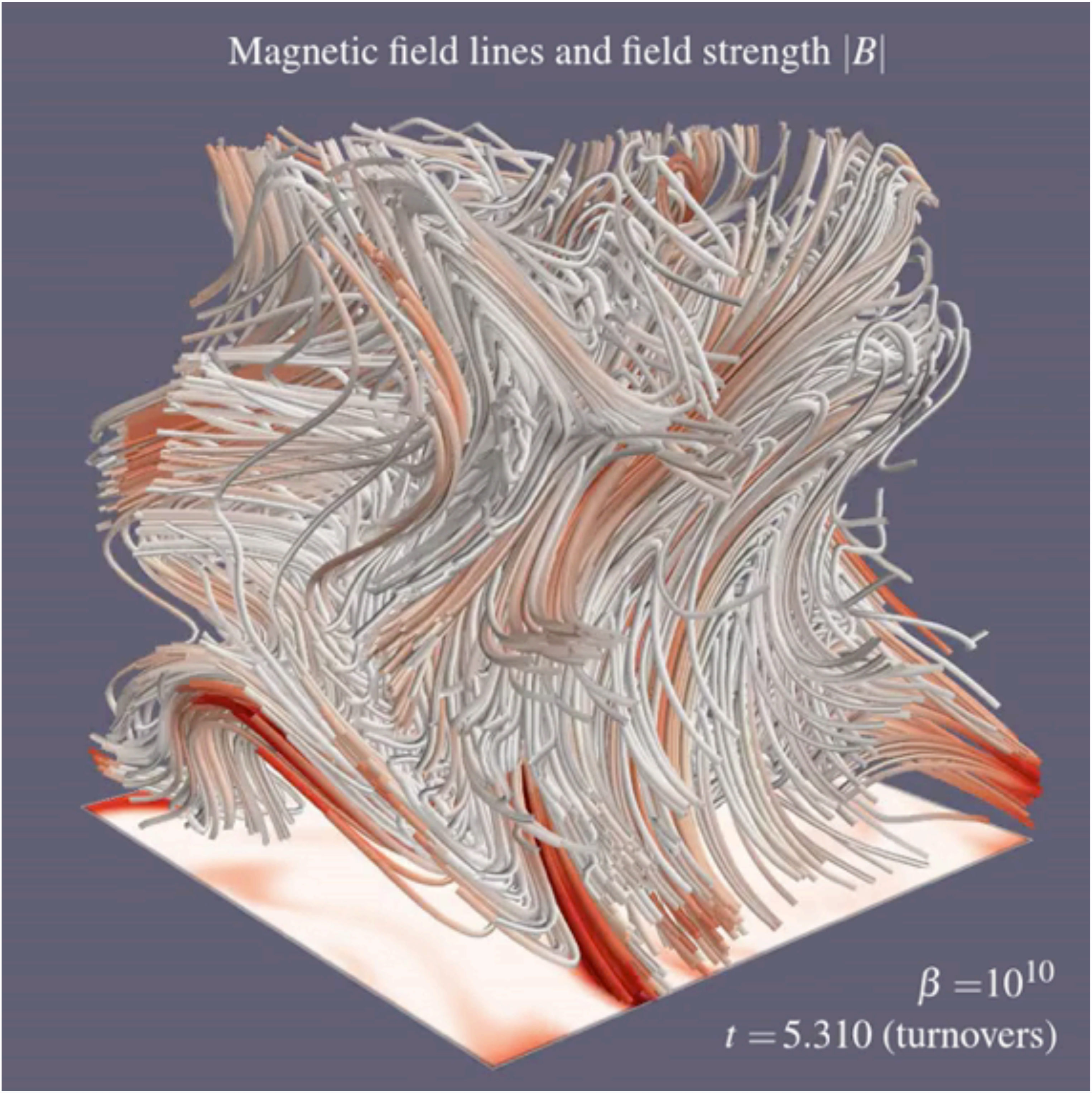
$$\beta = 10^{10}$$

$$\rho_i/L = 16$$

$$\mathrm{Rm} = \frac{u_{\mathrm{r.m.s.}}}{\eta k_f}$$

[Rincon et al., PNAS 2016]

Unmagnetized regime: growing case

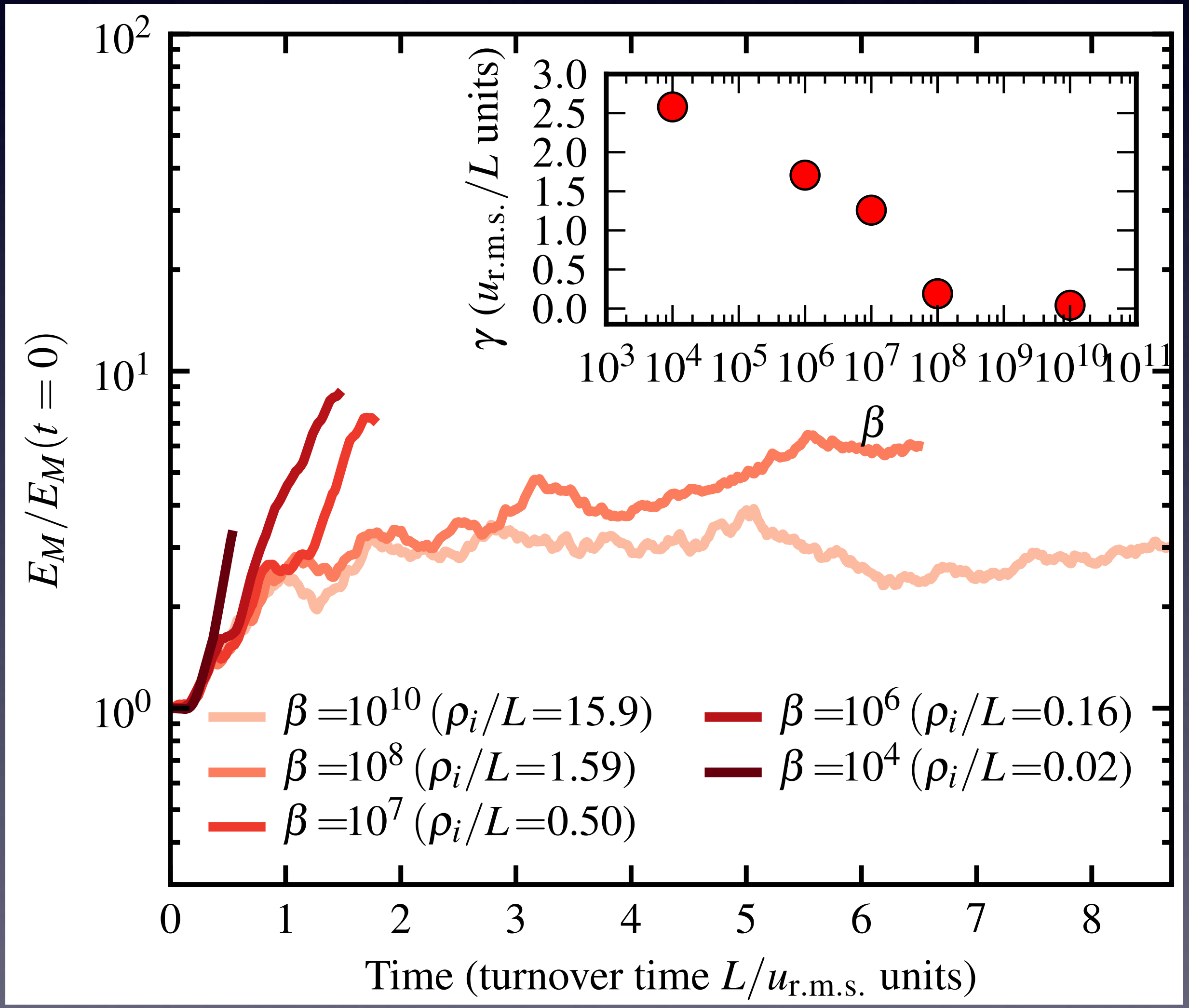


$\beta = 10^{10}$

$\rho_i/L \simeq 16$

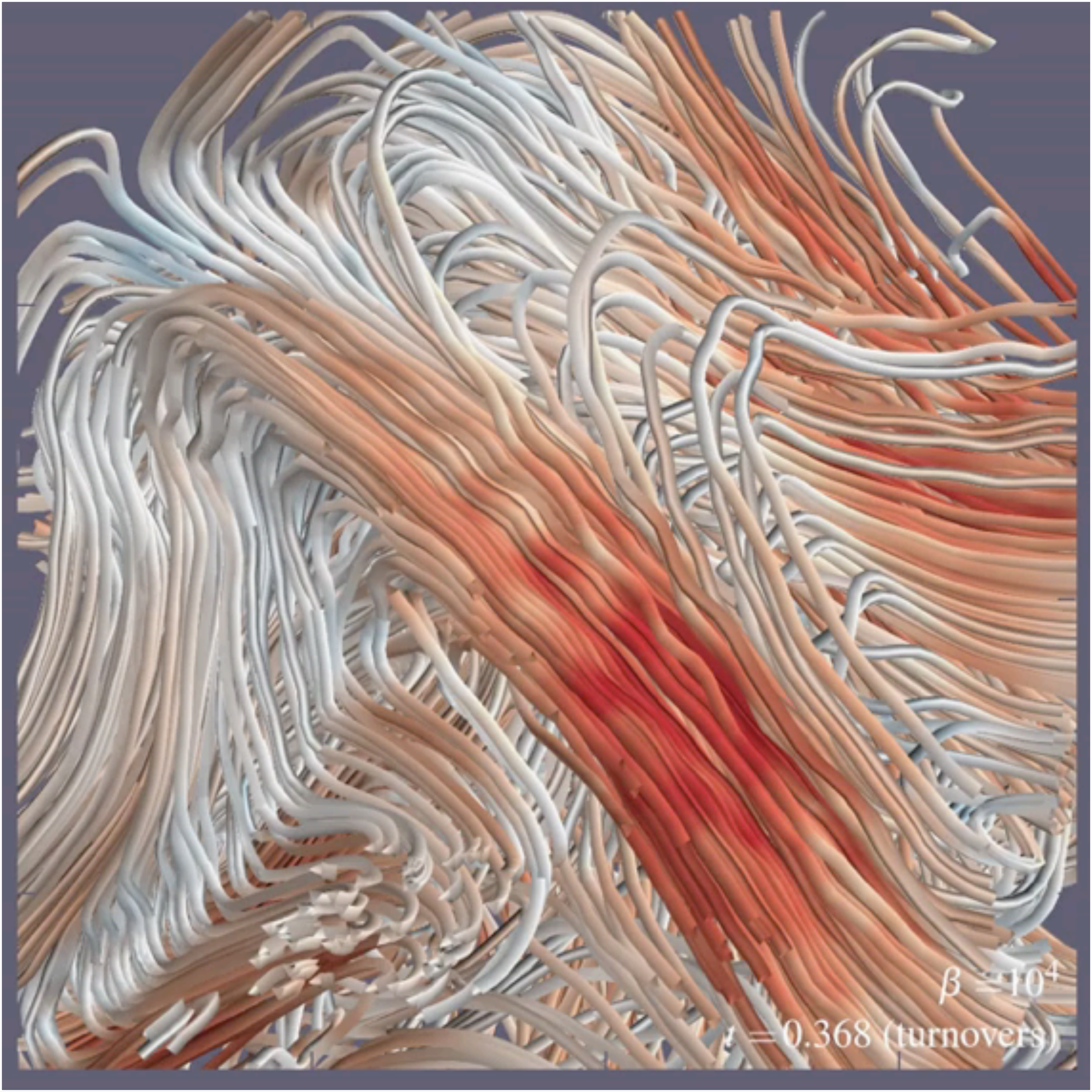
Exploring the magnetization transition

- Four simulations with same resistivity and input power, but different initial values of β



- Magnetic growth appears to self-accelerate

Magnetized regime

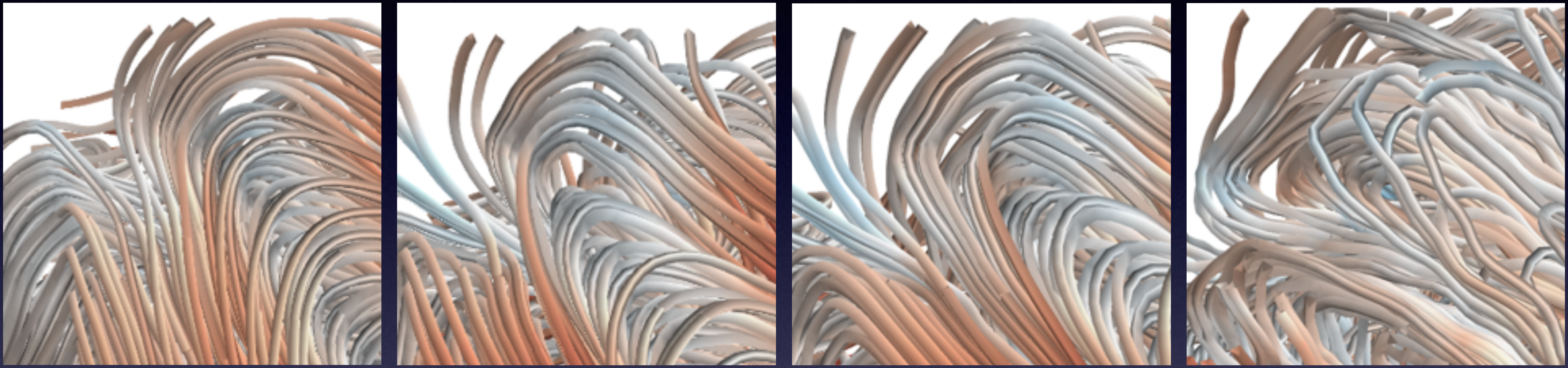


$\beta = 10^4$

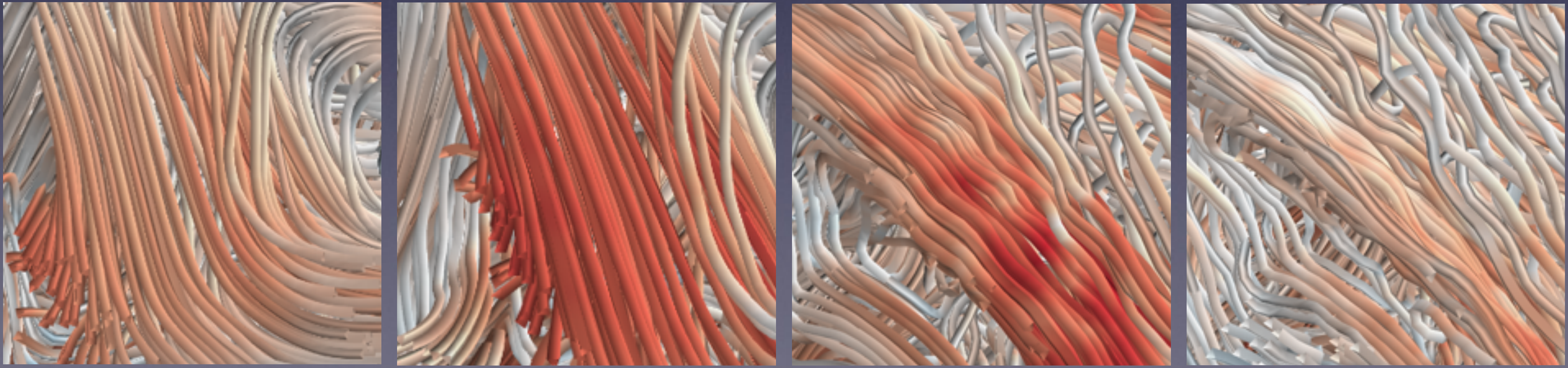
$\rho_i/L \simeq 0.02$

Magnetized regime

- Firehose instability in strong-field curvature regions



- Bubbly mirror fluctuations in field-stretching regions



Conclusions

Tomorrow’s fundamental theory challenges

- Turbulent large and small-scale dynamos
  - Unified, self-consistent nonlinear multiscale statistical dynamo theory
    - Requires physically justified closures
  - Description of asymptotic regimes (very high Re and Rm, low Pm, strong rotation)
- Interactions with different physical processes and geometrical effects
  - MHD instabilities combined to shear (magnetic buoyancy, MRI etc.)
  - Coherent structures (vortices, zonal flows, convection columns, tangent cylinders)
  - Plasma effects (batteries, pressure anisotropies, partial ionization etc.)
  - Reconnexion
- History of cosmic magnetism
  - from the pre-CMB era to stellar and planetary magnetic fields

Some good reads

- Books
  - Moffatt. *Magnetic field generation in electrically conducting fluids*, CUP (1978)
  - Zel'dovich, Ruzmaikin, Sokoloff. *Magnetic fields in astrophysics*, Gordon & Breach (1983)
  - Childress & Gilbert. *Stretch, twist, fold, the fast dynamo*, Springer (1995)
- Reviews and book chapters
  - Proctor. "Dynamo processes: the interaction of turbulence and magnetic fields", in *Stellar Astrophysical Fluid dynamics* (2003)
  - Diamond, Hughes & Kim. "Self-consistent mean-field electrodynamics in 2 and 3 dimensions", in *Fluid Dynamics and Dynamos in Astrophysics and Geophysics* (2005)
  - Brandenburg & Subramanian. "Astrophysical magnetic fields and nonlinear dynamo theory", Phys. Rep. (2005)
  - Schekochihin. "Turbulence and magnetic Fields in astrophysical plasmas", in *MHD: historic evolution and trends* (2007)
  - Brandenburg. "Advances in theory and simulations of large-scale dynamos", Space Sci. Rev. (2009)
  - Charbonneau. "Dynamo models of the solar cycle", Living Rev. Sol. Phys. (2010)
  - Tobias, Cattaneo & Boldyrev. "MHD dynamos and turbulence", in *Ten Chapters in Turbulence* (2013)
  - Ogilvie. "Lecture notes on astrophysical fluid dynamics". J. Plasma Phys. 82, 205820301 (2016)