

Grain temperature fluctuations: a key to infrared spectra

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Summary. Interstellar grains of radius less than $0.03\ \mu\text{m}$ are known to have a fluctuating temperature because their internal energy is comparable to the energy of UV and visible photons. Here, we give the exact resultant grain temperature distribution by studying the multiphoton absorption processes. This distribution satisfies an homogeneous integral equation which can be solved by an iteration technique.

These small grains generally emit at shorter wavelengths than big cold grains and therefore could help in the interpretation of the infrared spectra ($1\text{--}1000\ \mu\text{m}$) of various astrophysical objects.

Key words: infrared radiation – interstellar medium: dust and general

1. Introduction

Since the late sixties, when Greenberg (1968) made the first suggestion, a great deal of effort has been put into understanding the way grain temperature can evolve (Duley, 1973; Greenberg and Hong, 1974; Purcell, 1976; Aannestad and Kenyon, 1979; Draine and Anderson, 1985). The internal energy, at low temperature, of grains of radius less than about $0.03\ \mu\text{m}$ is comparable to the energy of a typical interstellar photon. The temperature of these grains fluctuates in time and is therefore not equal to what the energy balance would have said.

The solution of this problem may be the key to modelling the IR spectra of various astrophysical objects ranging from reflection nebulae (Sellgren et al., 1983; Sellgren, 1984), to the interstellar medium (Boulanger et al., 1985), the center of the Galaxy (Pajot et al., 1985), and whole galaxies (e.g. Roche and Aitken, 1985). Recently, this problem has generated a renewed interest in the astrophysical community because of the possible explanation by the mechanism of grain temperature fluctuations, of the strong IR emission features at 3.3 , 6.2 , 7.7 , 8.6 , $11.3\ \mu\text{m}$ (Léger and Puget, 1984; Puget et al., 1985). These features appear to be due to emission by small grains, actually great molecules identified with polycyclic aromatic hydrocarbons (PAH), that are transiently heated to high temperature by UV and visible photon absorption.

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The motivation for our approach comes from the fact that only for the smallest grains is the monophoton process (Purcell, 1976; Puget et al., 1985) an appropriate description, because the cooling time is short compared with the arrival rate for the harder photons responsible for the temperature fluctuations. On the other hand, the big grains have a large heat capacity, and therefore, their temperature does not fluctuate and is simply given by the energy balance between radiative emission and absorption. However, intermediate size grains (of order $3\text{--}30\ \text{nm}$), which have a cooling time and specific heat of intermediate value, cool sufficiently slowly that they may absorb several photons between complete cooling periods. The process of calculating the expected temperature distribution is then analogous to a single server queue, which has a characteristic competition between fatigue of the server and arrival rate in the queue.

We want here to give the exact solution of the multiphoton absorption process acting on grains. It can be calculated by an iteration procedure. Our treatment is therefore different from that of Draine and Anderson (1985) who used a Monte-Carlo method. In a forthcoming paper we will present the astrophysical applications, especially the IR spectrum, between 1 and $1000\ \mu\text{m}$, resulting from a mixture of grains and different incident radiation fields (preliminary results are given by Désert, 1985), and the consequences for chemistry on small grains.

In the following, we consider one grain embedded in an isotropic radiation field. After having described the elementary phenomena of emission and absorption, we derive by statistical arguments the evolution of the grain temperature. Then, we propose an iteration method for finding the stationary temperature distribution and give some examples. We show in the appendices the details of the needed eigenvalue argument and the correspondence between the Monte-Carlo methodology and our approach through the use of a stochastic differential equation method.

2. The elementary physical processes modifying the grain temperature

Only one variable is needed for the description of the grain state, namely its internal energy U . If in addition, we assume a continuum of energy levels for the grain, there is a strict correspondance between U and the temperature, the link being the grain heat capacity $C(T)$:

$$U = \int_0^T C(T') dT'. \quad (1)$$

Let us now describe the emission and absorption processes.

2.1. The discrete processes

2.1.1. The heating rate

We define $I(E)$ as the spectrum of the incident radiation (it can be expressed in unit of $\text{erg cm}^{-2} \text{s}^{-1} \text{sr}^{-1} \text{erg}^{-1}$) and $\sigma(E, a)$ as the absorption cross-section of a grain of radius a . The absorption rate of photons of energy between E and $E + dE$ is then:

$$J_1(E) dE = 4\pi \frac{I(E)}{E} \sigma(E, a) dE. \quad (2)$$

When a photon of energy E has just been absorbed by the grain, its energy is spread nearly instantaneously ($t \sim 10^{-8} \text{s}$, Duley, 1973) into the numerous grain modes, and the internal energy rises from U to $U + E$. Therefore, the transition rate by discrete heating from the state U_1 to the states between U_2 and $U_2 + dU_2$ can be written as:

$$J(U_1, U_2) dU_2 = J_1(U_2 - U_1) dU_2, \quad (3)$$

with $U_2 > U_1$.

2.1.2. The cooling rate

Like by the discrete heating previously discussed, the grain can cool by a jump process with a rate $G(U_1, U_2) dU_2$ emitting a photon of energy $U_1 - U_2$ when coming from the state U_1 . For our grain the Kirchhoff's law gives us:

$$G(U_1, U_2) dU_2 = 4\pi \frac{B(T_1, U_1 - U_2)}{U_1 - U_2} \sigma(U_1 - U_2, a) dU_2, \quad (4)$$

where $B(T_1, E)$ is the intensity of light [expressed in the same unit as $I(E)$] emitted at the energy E by a blackbody at a temperature T_1 [here corresponding to the internal energy U_1 by the formula (1)].

The rate of leaving the state U_1 for the state U_2 by these two discrete processes (3) and (4) is defined by:

$$\begin{aligned} L(U_1, U_2) &= J(U_1, U_2) \text{ for } U_1 < U_2, \\ &= G(U_1, U_2) \text{ for } U_1 > U_2. \end{aligned} \quad (5)$$

The total rate for leaving the state U_1 is defined as the sum:

$$L_0(U_1) = \int_0^\infty L(U_1, U_2) dU_2. \quad (6)$$

2.2. The continuous process

When grain heating or cooling occurs by radiation with an energy that is very small compared with the heat content of the grain, it is convenient to approximate this phenomenon by a continuous drift of the internal grain energy at a rate:

$$dU/dt = -g(U), \quad (7)$$

where g is a function of the state (cooling) or not (heating). To take into account all of the photons of small energy, we define an energy cut-off E_c that can be fixed by the need of a given calculation (precision, respective role of lines and continuum). Then the g function must be equal to:

$$g(U) = \int_0^{E_c} dE E (G(U, U - E) - J_1(E)). \quad (8)$$

Naturally, this modifies the discrete processes, defined by the L function (formula (5)), by reducing it to photons of energy higher than E_c . Other cooling or heating mechanisms such as exchanges

of particles or chemistry on the grain, can be simply added if necessary. As the g function generally increases with U , we can define the zero level of internal energy in formula (1) as the level for which g becomes positive.

3. Evolution equation for the internal energy distribution

The global process discussed here can be described by the *internal energy distribution* (IED):

$$P_{t_1, t_2}(U_1, U_2) dU_2 = \text{Prob}[\mathcal{U}_{t_2} = U_2 | \mathcal{U}_{t_1} = U_1] dU_2 \quad (9)$$

for two moments $t_1 < t_2$. The IED is the conditional probability for the random variable \mathcal{U}_{t_2} (the internal energy at time t_2), to be between U_2 and $U_2 + dU_2$, knowing that, at a preceding time t_1 , \mathcal{U}_{t_1} was equal to U_1 . Because the process of arriving to the state U_2 at time t_2 , starting from U_1 at time t_1 does not depend on the history of the grain before time t_1 , it is a Markov process (see e.g. Gardiner, 1983). Furthermore it depends only on the difference $t_2 - t_1$, so that we can define the probability distribution:

$$P_t = P_{t', t'+t}, \quad (10)$$

which is independent of t' (homogeneity). It satisfies the Chapman-Kolmogorov equation for any two times t_1 and t_2 :

$$P_{t_1+t_2} = P_{t_1} \circ P_{t_2}, \quad (11)$$

where the composition law is defined as:

$$(P_{t_1} \circ P_{t_2})(U_1, U_2) = \int_0^\infty dU P_{t_1}(U_1, U) P_{t_2}(U, U_2). \quad (12)$$

Equation (11), with the definition (12), means that the probability is conserved when summing over all the intermediate states U at the intermediate time t_1 .

The differential version of the Eq. (11) yields the evolution equation [second Kolmogorov equation, Gardiner (1983)]:

$$dP_t/dt = P_t \circ A, \quad (13)$$

where the evolution operator $A(U, U_2)$ is given by:

$$A(U, U_2) = \lim_{\Delta t \rightarrow 0} \frac{P_{\Delta t}(U, U_2) - \delta(U_2 - U)}{\Delta t}, \quad (14)$$

where δ is the Dirac function.

The calculation of A includes all the elementary physical processes discussed in Sect. 2. It is carried out by evaluating, for small Δt , the probability distribution:

$$\begin{aligned} P_{\Delta t}(U, U_2) \\ = \Delta t L(U, U_2) + (1 - L_0(U) \Delta t) \delta[U_2 - (U - g(U) \Delta t)] + O(\Delta t^2). \end{aligned} \quad (15)$$

With a probability given by the first term of this equation, the grain absorbs or emits a photon. With the complementary probability, the grain certainly cools to the state $U - g(U) \Delta t$, which gives the second term to first-order. Then, it is easy to show from (14) and (15) that A is:

$$A(U, U_2) = L(U, U_2) - L_0(U) \delta(U_2 - U) + g(U) \delta'(U_2 - U), \quad (16)$$

where δ' is the derivative of the Dirac function. Using Eqs. (13) and (16), the evolution equation can be written as:

$$\begin{aligned} dP_t(U_1, U_2)/dt = \int_0^\infty dU P_t(U_1, U) L(U, U_2) - L_0(U_2) P_t(U_1, U_2) \\ + d/dU [g(U) P_t(U_1, U)]|_{U=U_2}. \end{aligned} \quad (17)$$

4. The internal energy distribution as solution of an integral equation

It is clear that in any astrophysical situations, many photons are absorbed or emitted by each grain in astrophysical time unit of 1 Myr (for example, a graphite grain of 1 nm radius embedded in the interstellar radiation field of the solar neighborhood (Mathis et al., 1983) absorbs one photon per year) so that a population of grains of same properties reaches a *stationary* internal energy distribution (SIED). We here show that, indeed, this SIED exists and we describe it. To get a SIED, we must look for solutions satisfying the statistical equilibrium condition:

$$dP_t/dt = 0. \quad (18)$$

If a unique SIED exists, it cannot depend on the initial condition of the grain population, so that in applying Eq. (17) with (18), we can forget the starting energy level U_1 and look for a SIED $P(U)$ satisfying:

$$d[g(U)P(U)]/dU = L_0(U)P(U) - \int_0^\infty dU' P(U') L(U', U). \quad (19)$$

This can be interpreted as a detailed balance for the “flux of probability” $g(U)P(U)$. By integrating this equation over all energy and using Eq. (6) we find that $g(U)P(U)$ must go to zero when U becomes infinite, because $g(0)$ vanishes. Now, let us define:

$$\tau(U_1, U) = \int_U^{U_1} dU' L_0(U')/g(U') \quad (20)$$

as the “time” needed by the grain to cool down, from the state U_1 to the state U measured in the time unit defined by the inverse of the total rate of the discrete processes (τ is therefore dimensionless). Let us define the function:

$$H(U_1) = g(U_1) P(U_1) e^{-\tau(U_1, U_2)}, \quad (21)$$

where U_2 is a constant (which will drop at the end of the calculation). By changing U in U_1 in Eq. (19), we obtain that the function H satisfies the equation:

$$dH(U_1)/dU_1 = -e^{-\tau(U_1, U_2)} \int_0^\infty dU' P(U') L(U', U_1), \quad (22)$$

which is readily integrated and gives the SIED as the solution of the integral equation:

$$P(U) = \int_U^\infty dU_1 \frac{e^{-\tau(U_1, U)}}{g(U)} \int_0^\infty dU' P(U') L(U', U_1). \quad (23)$$

The probability of finding the grain in a state U is calculated as the sum of all the probabilities of starting from any state U' , then going to *any* intermediate “excited” state $U_1 > U$ by the discrete processes, and, finally, decaying by the continuous process to the state U . The exponential term is directly related to the Poisson statistics. Figure 1 shows schematically the collective processes described in this section.

We can rearrange this last expression by defining the transition kernel K :

$$K(U', U) = \int_U^\infty dU_1 e^{-\tau(U_1, U)} L(U', U_1)/g(U). \quad (24)$$

From Eqs.(23) and (24), the SIED satisfies the following homogeneous Fredholm integral equation of the second type (Heywood and Fréchet, 1912):

$$P(U) = \int_0^\infty dU' P(U') K(U', U). \quad (25)$$

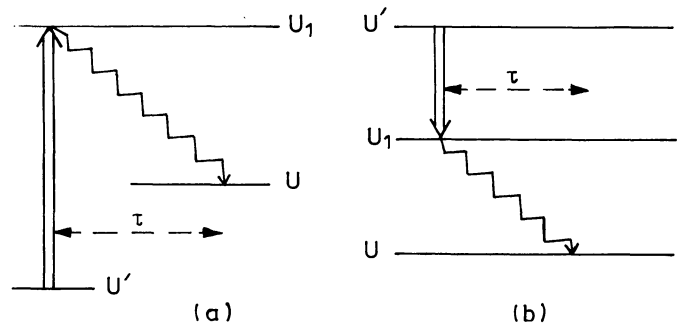


Fig. 1 a and b. Schematic representation of absorption (a) and emission (b) processes for change of the grain internal energy. Energy states U, U_1, U_2 are defined in the text. Discrete absorption or emission is represented by a solid line. Continuous cooling is represented by a wavy line. The “cooling time” for the level U is τ

In Appendix A, we give the mathematical proof of the existence of the SIED. Appendix B gives the link of this solution with the Monte-Carlo approach (Langevin equation).

5. Numerical approach and results

The integral Eq. (23) means that the SIED is stable against the absorption or emission of one photon. This fact suggests that starting with a guessed distribution P_g , the iteration of the kernel K upon P_g may converge to the SIED. Actually, we have naturally discretized the levels of energy and then iterated the kernel itself (in order to minimize the numerical errors):

$$K^{(0)} = K, \quad (26)$$

and, [with the definition (12)]:

$$K^{(n)} = K^{(n-1)} \circ K^{(n-1)}, \quad (27)$$

and applied $K^{(n)}$ (n being large enough) to P_g , which may be taken as a function peaked at the equilibrium temperature (that is, we start with a grain at a known temperature):

$$P(U) = \int_0^\infty dU' K^{(n)}(U', U) P_g(U'). \quad (28)$$

Starting from a δ function, each iteration of the kernel corresponds to multiplying the number of photon arrivals by a factor of 2. Indeed, the first iteration corresponds to looking only at processes where the cooling time to arrive at the internal energy U is short compared to the typical time between photon absorptions [monophoton approximation (Purcell, 1976; Puget et al., 1985)]. Further iterations take into account the effect of processes in which $2, 2^2, \dots, 2^n$ photons are absorbed before cooling to the state U . In appendix C, we discuss the mathematical proof of convergence of the iteration procedure. Previous studies, especially Draine and Anderson (1985) and Yorke (1985), have employed the temperature as the independent variable for probability distribution. As we noted in the introduction, there is a one-to-one correspondence between the internal energy and the temperature [see formule (1)]. Thus, we have the freedom to compute the SIED for one grain in the unit which is natural to the absorber itself, that is the internal energy, and then to transform this into “observational” units, namely the temperature.

For numerical calculations, we have considered only graphite grains. Graphite heat capacity is taken from American Institute of

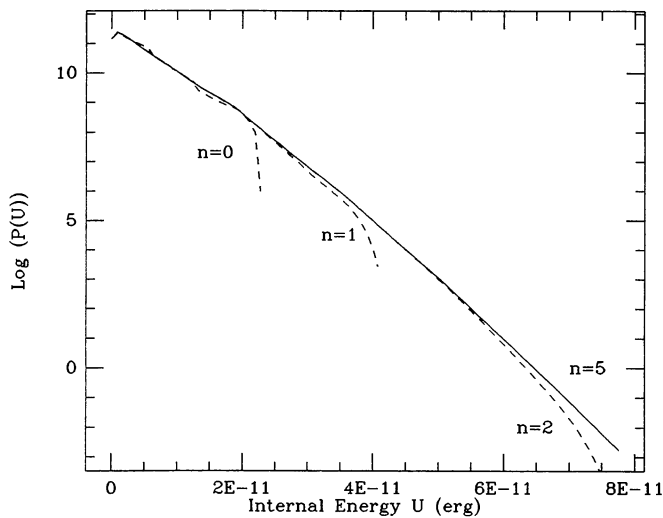


Fig. 2. Sample calculation of the iteration process. The differential internal energy distribution $P(U)$ as a function of U (in ergs) is plotted for several iteration number n . Parameters used: graphite grain, radius of $0.01 \mu\text{m}$, standard interstellar radiation field, matrix of iteration: 100×100

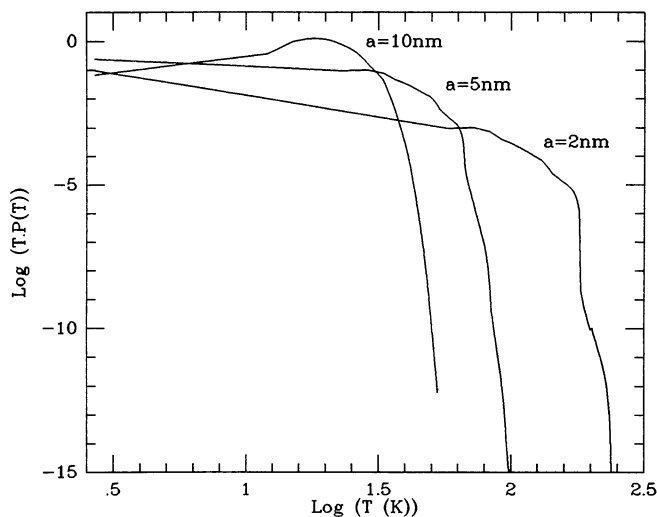


Fig. 3. Temperature distribution $[T, T \cdot P(T)]$ corresponding to graphite grains of different radius: 10, 5, 2 nm, embedded in the interstellar radiation field

Physics Handbook (1969) and it is proportional to the square of the temperature in the temperature range of interest (DeSorbo and Tyler, 1953), a behavior quite different from the Debye approximation, due to the layer structure of graphite (Ziman, 1962). The emissivities have been taken from Draine and Lee (1984), and the grains are embedded in the solar neighborhood interstellar radiation field (Mathis et al., 1983).

For a grain radius of 10 nm, as shown in Fig. 2, the internal energy distribution is indeed numerically converging quickly to a SIED within less than 5 iterations. For smaller grains, the convergence is faster. For much bigger grains, due to the high heat capacity, the temperature fluctuations remain low. Using the correspondence given by formula (1), the temperature distributions of grains of different sizes are displayed in Fig. 3. Figure 4

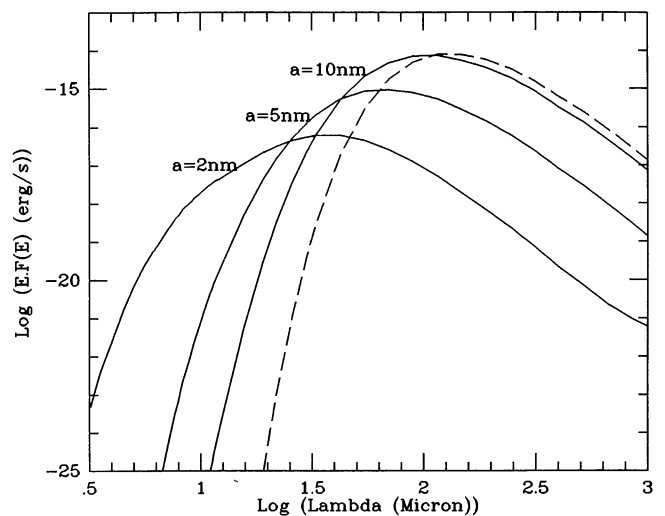


Fig. 4. Infrared spectra emitted by grains of same sizes as in Fig. 3. Dashed line corresponds to the emission of the 10 nm grain if it were always at the equilibrium temperature

shows the resultant infrared spectra for these grains, calculated with:

$$F(E) = \int_0^{\infty} dU P(U) E G(U, U-E). \quad (29)$$

Our method can be used to compute infrared spectra of grain population with a wide size distribution without many iterations.

These results are slightly different from previous results, Draine and Anderson (1985) obtained by Monte-Carlo simulations, because we use a different heat capacity, an important function when dealing with temperature fluctuations. The accuracy we get with our method, can be achieved with Monte-Carlo simulations only with a large number of absorbed photons. Indeed, for a 1 nm graphite grain, 10^6 absorbed photons are needed to account for the high temperature tail of the temperature distribution which is produced by the rare multiphoton absorption events. This is especially critical for grain evaporation calculation since the evaporation rate steeply increases with temperature (Puget et al., 1985).

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Appendix A: existence of the SIED

From the mathematical viewpoint, the distribution P can be seen as an eigenvector of the linear operator \mathcal{K} defined by the kernel K , with an eigenvalue of unity. To show that unity truly belongs to the spectrum of \mathcal{K} will prove that there exists a solution for the SIED.

First, we remark that the kernel, defined by Eq. (24), satisfies the differential equation:

$$K(U', U) L_0(U) = L(U', U) + d[g(U), K(U', U)]/dU. \quad (A1)$$

By integrating the kernel K on U we find that:

$$\int_{U_2}^{\infty} dU K(U', U) L_0(U) = \int_{U_2}^{\infty} dU L(U', U) - g(U_2) K(U', U_2) + \lim_{U \rightarrow \infty} [g(U) K(U', U)]. \quad (\text{A2})$$

The limit vanishes because we can assume that there are no photons of energy higher than a specified finite value [see formula (24)]. By letting U_2 go to zero and using the definition of L_0 [see formula (6)], we obtain:

$$\int_0^{\infty} dU K(U', U) L_0(U) = L_0(U'). \quad (\text{A3})$$

This equation simply means that L_0 is an eigenvector of the transposed operator ${}^t\mathcal{K}$ with an eigenvalue of unity. As the spectrum of ${}^t\mathcal{K}$ is identical to that of \mathcal{K} , we deduce that a normalisable solution exists for the SIED.

Appendix B: Langevin equation approach

Consider the energy balance equation of the form:

$$C(T) dT/dt = -g(T) + H_e, \quad (\text{B1})$$

where T is the temperature, $g(T)$ is the continuous cooling function and H_e is the heating function. Assume that the cooling, being radiative, has the form:

$$g(T) = Q T^m,$$

where Q is a constant and m is dependent on the absorption properties of the material. The specific heat will also be generalized ($C = C_0 T^n$) (see e.g. Ziman, 1962), so that the final form of the equation can be written as:

$$dy/dt = -A y^\alpha + \gamma(t; \Delta t), \quad (\text{B3})$$

where $y = T^{n+1}$, $A = (n+1)Q/C_0$ and $\alpha = m/(n+1)$. This is a nonlinear, first order Langevin equation, if we assume that the heating function $\gamma = (n+1)H_e/C_0$, which is independent of temperature, is a random function of time. Specifically, assume that the intervals of arrival of heating photons are uncorrelated, and that the distribution function for the heating can be taken as a Wiener or a Poisson process. This assumption has been used often in the literature, specifically by Purcell (1976) and Aannestad and Kenyon (1979).

The equation for the temperature function is:

$$\Delta y = -A \int_0^t y^\alpha(t') dt' + \int_0^t \gamma(t'; \Delta t) dt'. \quad (\text{B4})$$

Now, the formal solution of this equation can be achieved in two separate ways (see e.g. van Kampen, 1981). One is to take the iterative solution of a nonlinear integral equation. The alternative, which we shall exploit here, is to assume that the heating is primarily continuous, by the infrared (“soft”) photons, and that the hotter end of the spectrum is a perturbation of the heating (though it is actually the principal):

$$\gamma(t, \Delta t) = \gamma_0 + \gamma_1(\Delta t) \quad (\text{B5})$$

and, to zeroth order, by using the differential equation (B3), we obtain:

$$y_0 = \left(\frac{\gamma_0}{A} \right)^{1/\alpha} \quad (\text{B6})$$

Expanding the Eq. (B3) to first order gives:

$$\frac{dy_1}{dt} = -\alpha A y_0^{\alpha-1} y_1 + \gamma_1(\Delta t) = -\frac{y_1}{\tau} + \gamma_1(\Delta t) \quad (\text{B7})$$

for the temperature function. The moments for this function can be obtained as follows. Assuming that the mean of γ_1 is Γ_1 , take the mean:

$$\langle y_1(t) \rangle = \frac{\Gamma_1}{\alpha} \left(\frac{C_0}{(n+1)Q} \right)^{1/\alpha} \gamma_0^{(1-\alpha)/\alpha} (1 - e^{-t/\tau}) \quad (\text{B8})$$

which is simply related to the Ornstein-Uhlenbeck equation (van Kampen 1981; Gardiner, 1983). This solution merely indicates the way the multiphoton process produces a change in the distribution function for the grain temperature, but it is physically directly associated with the processes previously discussed in the literature and the ancillary procedure for the derivation of the Chapman-Kolmogorov equation.

Appendix C: convergence of the iteration procedure

Appendix A demonstrates that unity is an eigenvalue of the operator \mathcal{K} . Because this operator has only positive values, one can show, by an argument of convexity, that it necessarily has an eigenvector $V(U)$ that is positive for every U . We want to show that this eigenvector cannot differ from the SIED $P(U)$ we are looking for. For any eigenvector $R(U)$ associated with an eigenvalue r , we calculate the following expression by inversion of the integration order:

$$\iint dU' dU R(U') K(U', U) L_0(U),$$

and get the following equality [using formula (A3)]

$$\int_0^{\infty} dU' R(U') L_0(U') = r \int_0^{\infty} dU R(U) L_0(U). \quad (\text{C1})$$

Thus, as L_0 is always positive (see formula (6)), unless the eigenvalue r is unity, the eigenvector R is somewhere negative, which is not acceptable for a probability distribution.

A Markov process, by definition, has no explicit dependence on the “history” of the stochastic variable. Furthermore, it depends at time $t_i > t_{i-1}$ only on the state t_{i-1} and the transition probability. Thus, the process is homogeneous, in that it is independent of the starting point (see formula (10)). For our purpose, this means that if we have an initial distribution P_g , then:

$$P^{(n)} = K^{(n)} P_g = K^{(n-m)} P^{(m)} \xrightarrow{n \rightarrow \infty} P, \quad (\text{C2})$$

so that if n is large enough, the distribution is stationary. The general proof of the convergence of the iteration technique is provided by Doob (1953) and Gihman and Skorohod (1980), for finite Markov chains (a good approximation for our problem), that there is always at least one set of stationary absolute probabilities. We can do no more than emphasize how basic this property is to the Markov process than to quote Doob’s note to the chapter (p. 628): “The fundamental result in the case of finitely many states ..., goes back to Markov (1906) and has been

rediscovered frequently". Finally, a by-product of the convergent iteration procedure is the uniqueness of the solution P .

References

- American Institute of Physics Handbook: 1969, McGraw-Hill, New York, pp. 4–105
- Aannestad, P.A., Kenyon, S.J.: 1979, *Astrophys. J.* **230**, 771
- Boulanger, F., Baud, B., van Albada, G.D.: 1985, *Astron. Astrophys.* **144**, L9
- Désert, F.X.: 1985, 1st IRAS Symposium *Light on Dark Matter*, ed. F. P., Reidel, Dordrecht Israel
- DeSorbo, W., Tyler, W. W.: 1953, *J. Chem. Phys.* **21**, 1660
- Doob, J.L.: 1953, *Stochastic Processes*, Wiley, New York, p. 172ff
- Draine, B.T., Lee, H.M.: 1984, *Astrophys. J.* **285**, 89
- Draine, B.T., Anderson, N.: 1985, *Astrophys. J.* **292**, 494
- Duley, W. W.: 1973, *Astrophys. Space Sci.* **23**, 43
- Gardiner, C. W.: 1983, *Handbook of Stochastic Methods*, Springer, Berlin-Heidelberg-Hew York
- Gihman, I.I., Skorohod, A.V.: 1980, *The Theory of Stochastic Processes. I.*, Springer, Berlin, Heidelberg, New York, p. 75
- Greenberg, J.M.: 1968, in *Stars and Stellar Systems*, Vol. 7, Nebulae and Interstellar Matter, eds. B.M. Middlehurst, L.H., Chicago University Press, p.221
- Greenberg, J.M., Hong, S.S.: 1974, in *Galactic Radio Astronomy*, IAU Symp. eds. F.J. Kerr, S.C. Simonson, Reidel, Dordrecht p. 155
- Heywood, H. B., Fréchet M.: 1912, *L'équation de Fredholm et ses applications à la physique mathématique*, Herman, Paris
- Léger, A., Puget, J.L.: 1984, *Astron. Astrophys.* **137**, L5
- Mathis, J.S., Mezger, P. G., Panagia, N.: 1983, *Astron. Astrophys.* **126**, 212
- Pajot, F., Boissé, P., Gispert, R., Lamarre, J. M., Puget, J. L., Serra G.: 1985, *Astron. Astrophys.* (in press)
- Puget, J.L., Léger, A., Boulanger, F.: 1985, *Astron. Astrophys.* **142**, L19
- Purcell, E. M.: 1976, *Astrophys. J.* **206**, 685
- Roche, P.F., Aitken, D.K.: 1985, *Monthly Notices Roy. Astron. Soc.* **213**, 789
- Sellgren, K., Werner, M. W., Dinerstein H. L.: 1983, *Astrophys. J. Letters* **271**, L13
- Sellgren, K.: 1984, *Astrophys. J.* **277**, 623
- van Kampen, N.G.: 1981, *Stochastic Processes in Physics and Chemistry*, North-Holland, Amsterdam
- Yorke, H.: 1985, in *Birth and Infancy of Stars: XLI Les Houches Summer School (1983)*, eds. R., Lucas, A. Omont, R. Stora, North-Holland
- Ziman, J.: 1962, *Electrons and phonons*, Oxford University Press, Oxford