Notes on the Boussinesq Approximation

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Sound waves are often a nuisance in many calculations. You might be interested in buoyancy and convective stability, or in Alfvén-wave turbulence in a high-β plasma, or in the magnetorotational instability. For these situations, sound waves play a rather boring role, and often serve only to make the algebra more tedious (or, in the case of a code, the timestep more restrictive). There is something called the Boussinesq Approximation, which rigorously filters out sound waves. These notes are meant as a supplement to AST521, and are designed to teach you about this approximation and how to use it.

Rather than give a precise def$^{\underline{n}}$ of the Boussinesq limit, let's do a problem and see what kind of extraneous information is involved and which we may want to avoid. Consider a stratified atmosphere in hydrostatic equilibrium; the pressure gradient $+\vec{\nabla}P = +\frac{dP}{dz}\hat{z}$ balances gravity $+\rho\vec{g} = -\rho g\hat{z}$:

$$-\frac{1}{\rho}\frac{dP}{dz} = g = \text{constant}.$$

(Let us ignore magnetic fields here.) We allow small perturbations about this equilibrium:

$$\rho = \rho_0(z) + \delta\rho \qquad P = P_0(z) + \delta P \qquad \vec{u} = \emptyset + \vec{\delta u}$$

so that the linearized hydrodynamic equations read

(1) $$\frac{\partial}{\partial t}\delta\rho + \rho_0 \vec{\nabla}\cdot\vec{\delta u} + \vec{\delta u}\cdot\vec{\nabla}\rho_0 = 0,$$

(2) $$\rho_0 \frac{\partial \vec{\delta u}}{\partial t} = -\vec{\nabla}\delta P - \delta\rho\, g\hat{z}.$$

We'll need to specify the pressure perturbation δP in terms of the density fluctuation, and for that we use the entropy equation:

$$\frac{P}{\gamma-1}\frac{D}{Dt}\ln\frac{P}{\rho^\gamma} = 0,$$

which, when linearized, becomes

(3) $$\frac{\partial}{\partial t}\left(\frac{\delta P}{P_0} - \gamma\frac{\delta\rho}{\rho_0}\right) + \vec{\delta u}\cdot\vec{\nabla}\ln P_0\rho_0^{-\gamma} = 0.$$

Let us also introduce $a^2 \equiv \frac{\gamma P_0}{\rho_0}$ as the adiabatic sound speed. Solutions to (1)–(3) are proportional to $e^{-i\omega t}$:

$$-i\omega \frac{\delta\rho}{\rho} + \vec{\nabla}\cdot\vec{\delta u} + \delta u_z \frac{d\ln\rho}{dz} = 0,$$

$$-i\omega \vec{\delta u} = -\frac{1}{\rho}\vec{\nabla}\delta P - \frac{\delta\rho}{\rho} g\hat{z},$$

$$-i\omega\left(\frac{\delta P}{P} - \gamma\frac{\delta\rho}{\rho}\right) + \delta u_z \frac{d\ln P\rho^{-\gamma}}{dz} = 0,$$

where I've dropped the "0" subscripts for notational ease. In general, we cannot Fourier transform these equations in z, because the coefficients in front of the perturbed quantities are z-dependent. But we can do so in the horizontal (say, x) direction:

$$-i\omega\frac{\delta\rho}{\rho} + ik_x \delta u_x + \frac{d\delta u_z}{dz} + \delta u_z \frac{d\ln\rho}{dz} = 0,$$

$$-i\omega\,\delta u_x = -ik_x \frac{\delta P}{\rho},$$

$$-i\omega\,\delta u_z = -\frac{1}{\rho}\frac{d\delta P}{dz} - \frac{\delta\rho}{\rho} g,$$

$$-i\omega\left(\frac{\delta P}{P} - \gamma\frac{\delta\rho}{\rho}\right) + \delta u_z \frac{d\ln P\rho^{-\gamma}}{dz} = 0,$$

where now the fluctuations are z-dependent Fourier amplitudes. Denoting $\vec{\delta u} = -i\omega\vec{\xi}$, we have

(a) $\frac{\delta\rho}{\rho} + ik_x\xi_x + \xi_z' + \xi_z \frac{d\ln\rho}{dz} = 0$

(b) $-\omega^2\xi_x = -ik_x \frac{\delta P}{\rho}$

(c) $-\omega^2\xi_z = -\frac{1}{\rho}\frac{d}{dz}\delta P - \frac{\delta\rho}{\rho} g$

(d) $\frac{\delta P}{P} = \gamma\frac{\delta\rho}{\rho} - \xi_z \frac{d\ln P\rho^{-\gamma}}{dz}$

(b) and (d) $\rightarrow$ $-\omega^2\xi_x = -ik_x\left(\frac{P}{\rho}\right)\left[\gamma\frac{\delta\rho}{\rho} - \xi_z\frac{d\ln P\rho^{-\gamma}}{dz}\right]$

and (a) $\rightarrow$ $-\omega^2\xi_x = +ik_x\frac{P}{\rho}\gamma\left[+ik_x\xi_x + \xi_z' + \xi_z\frac{d\ln\rho}{dz}\right]$

$+ik_x\frac{P}{\rho}\xi_z\frac{d\ln P\rho^{-\gamma}}{dz}$

$\Rightarrow (-\omega^2 + k_x^2a^2)\xi_x = ik_x a^2\xi_z' + \frac{ik_x a^2}{\gamma}\frac{d\ln P}{dz}\xi_z$

where $a^2 \equiv \gamma P/\rho$.

Note: $g = -\frac{a^2}{\gamma}\frac{d\ln P}{dz}$, so this is

$$\#\ \boxed{(-\omega^2 + k_x^2a^2)\xi_x = ik_x a^2\xi_z' - ik_x\xi_z g}$$

(a) $\Rightarrow$ $\frac{\delta\rho}{\rho} = -\xi_z' - \xi_z\frac{d\ln\rho}{dz} - \frac{ik_x\left[ik_x a^2\xi_z' - ik_x\xi_z g\right]}{-\omega^2 + k_x^2a^2}$

$$\Rightarrow \boxed{\frac{\delta\rho}{\rho} = \frac{\omega^2 \xi_z' + \left[(\omega^2 - k_x^2 a^2)\frac{d\ln\rho}{dz} - k_x^2 g\right]\xi_z}{k_x^2 a^2 - \omega^2}} \quad ©$$

$$\Rightarrow \boxed{\frac{\delta P}{P}} = \frac{1}{k_x^2 a^2 - \omega^2}\left[\gamma\omega^2 \xi_z' + \gamma\left((\omega^2 - k_x^2 a^2)\frac{d\ln\rho}{dz} - k_x^2 g\right)\xi_z + (\omega^2 - k_x^2 a^2)\frac{d\ln P\rho^{-\gamma}}{dz}\xi_z\right]$$

$$= \frac{1}{k_x^2 a^2 - \omega^2}\left[\gamma\omega^2 \xi_z' + \omega^2 \frac{d\ln P}{dz}\xi_z\right]$$

$$\boxed{= \frac{\omega^2}{k_x^2 a^2 - \omega^2}\left[\gamma \xi_z' + \frac{d\ln P}{dz}\xi_z\right]} \quad ☆$$

into (c):

$$+\omega^2 \xi_z = +g\left[\frac{\omega^2 \xi_z' + (\omega^2 - k_x^2 a^2)\frac{d\ln\rho}{dz}\xi_z - k_x^2 g \xi_z}{k_x^2 a^2 - \omega^2}\right]$$

$$+ \frac{1}{\rho}\frac{d}{dz}\left[\frac{\omega^2 P}{k_x^2 a^2 - \omega^2}\left(\xi_z' \gamma + \xi_z \frac{d\ln P}{dz}\right)\right]$$

$$= \frac{\omega^2}{\rho}\frac{dP}{dz}\frac{\left(\xi_z'\gamma + \xi_z\frac{d\ln P}{dz}\right)}{k_x^2 a^2 - \omega^2} + \frac{P}{\rho}\frac{\omega^2}{k_x^2 a^2 - \omega^2}\left(\gamma\xi_z'' + \xi_z'\frac{d\ln P}{dz} + \xi_z\frac{d^2\ln P}{dz^2}\right)$$

$$- \frac{\omega^2 P}{\rho}\frac{k_x^2}{(k_x^2 a^2 - \omega^2)^2}\frac{da^2}{dz}\left(\xi_z'\gamma + \xi_z\frac{d\ln P}{dz}\right)$$

$$\Rightarrow \quad \omega^2 \xi_z = \frac{g}{k_x^2 a^2 - \omega^2}\left[\omega^2 \xi_z' - k_x^2 g \xi_z + \left(\omega^2 - k_x^2 a^2\right)\frac{d\ln\rho}{dz}\xi_z\right]$$

$$+ \frac{\omega^2}{k_x^2 a^2 - \omega^2}(-g)\left[\xi_z' \gamma + \xi_z \frac{d\ln P}{dz}\right]$$

$$+ \left(\frac{a^2}{\gamma}\right)\frac{\omega^2}{k_x^2 a^2 - \omega^2}\left[\xi_z'' \gamma + \xi_z' \frac{d\ln P}{dz} + \xi_z \frac{g\gamma}{a^2}\frac{d\ln T}{dz}\right]$$

$$- \frac{\omega^2}{\gamma}\frac{a^2 k_x^2 a^2}{\left(k_x^2 a^2 - \omega^2\right)^2}\frac{d\ln T}{dz}\left[\xi_z' \gamma + \xi_z \frac{d\ln P}{dz}\right]$$

Mult. by $\dfrac{k_x^2 a^2 - \omega^2}{\omega^2 a^2}$ and group:

ξ_z'' : 1.

$$\xi_z' : \quad \cancel{\frac{g}{a^2}} - \frac{g}{a^2}\gamma + \frac{1}{\cancel{\gamma}}\cancel{\frac{d\ln P}{dz}} - \frac{k_x^2 a^2}{\cancel{\gamma}}\frac{1}{\left(k_x^2 a^2 - \omega^2\right)}\frac{d\ln T}{dz}\cancel{\gamma}$$

$$= \frac{d\ln P}{dz} - \left(\frac{k_x^2 a^2}{k_x^2 a^2 - \omega^2}\right)\frac{d\ln T}{dz} = \frac{(d\ln P/dz)}{\left(k_x^2 a^2 - \omega^2\right)}\left[k_x^2 a^2 - \omega^2 - k_x^2 a^2 \frac{d\ln T}{d\ln P}\right]$$

$$= \frac{d\ln P/dz}{k_x^2 a^2 - \omega^2}\left[-\omega^2 + k_x^2 a^2 \frac{d\ln\rho}{d\ln P}\right]$$

$$= \frac{\omega^2 \dfrac{d\ln P}{dz} - k_x^2 a^2 \dfrac{d\ln\rho}{dz}}{\omega^2 - k_x^2 a^2}$$

$$\xi_z: \; -\frac{(k_x^2a^2-\omega^2)}{a^2} - \frac{k_x^2 g^2}{\omega^2 a^2} - g\frac{d\ln\rho}{dz}\frac{k_x^2a^2 \cancel{-\omega^2}}{\omega^2a^2}$$

$$\cancel{-\frac{g}{a^2}\frac{d\ln P}{dz}} + \cancel{\frac{a^2}{\gamma}\frac{1}{g^2}\frac{g\gamma}{a^2}\frac{d\ln T}{dz}} - \frac{k_x^2a^2}{\gamma(k_x^2a^2-\omega^2)}\frac{d\ln T}{dz}\frac{d\ln P}{dz}$$

$$= \frac{-1}{k_x^2a^2-\omega^2}\Big[+\frac{k_x^2a^2}{\gamma}\frac{d\ln T}{dz}\frac{d\ln P}{dz} + \frac{g k_x^2}{\omega^2}\frac{d\ln\rho}{dz}(k_x^2a^2-\omega^2) + \frac{k_x^2 g^2}{a^2\omega^2}(k_x^2a^2-\omega^2) + \frac{(k_x^2a^2-\omega^2)^2}{a^2}\Big]$$

$$= \frac{1}{\omega^2-k_x^2a^2}\Big[\frac{\omega^4}{a^2} - 2\omega^2k_x^2 + k_x^4a^2 - \frac{k_x^2g^2}{a^2} \cancel{- g k_x^2\frac{d\ln\rho}{dz}} + \underbrace{\frac{k_x^2a^2}{\gamma}\frac{d\ln T}{dz}\frac{d\ln P}{dz}}_{-k_x^2 g\left(\frac{d\ln P}{dz} - \cancel{\frac{d\ln\rho}{dz}}\right)} + \frac{g k_x^4 a^2}{\omega^2}\frac{d\ln\rho}{dz} + \frac{k_x^4 \cancel{a^2} g^2}{\cancel{a^2}\,\omega^2}\Big]$$

$$= \frac{1}{\omega^2-k_x^2a^2}\Big[\frac{\omega^4}{a^2} - 2\omega^2k_x^2 + k_x^4a^2 \underbrace{- \frac{k_x^2g^2}{a^2} - k_x^2 g\frac{d\ln P}{dz}}_{-k_x^2 g \frac{d\ln P}{dz}\left(1-\frac{1}{\gamma}\right)} + g\frac{k_x^4a^2}{\omega^2}\frac{d\ln\rho}{dz} + \frac{k_x^4g^2}{\omega^2}\Big]$$

So,

$$\boxed{\xi_z'' + \xi_z'\left[\frac{\omega^2\frac{d\ln P}{dz} - k_x^2a^2\frac{d\ln\rho}{dz}}{\omega^2-k_x^2a^2}\right] + \xi_z\left[\;\cdots\;\right] = 0.}$$

This is UGLY!!! And we can't solve it analytically anyhow. It's just a stratified fluid — why is it so complicated?! The reason is twofold: (1) this equation mixes up buoyancy and sound waves — distinct physical effects; and (2) the sound and buoyancy frequencies are functions of height. Let's fix this by adopting an <u>ordering</u>: let $\frac{d\xi_z}{dz} \sim ik_z\xi_z = ik_zH\left(\frac{\xi_z}{H}\right) >> \frac{\xi_z}{H}$,

where $H \equiv \left|\frac{dz}{d\ln P}\right| \sim \left|\frac{dz}{d\ln\rho}\right|$. In other words, we assume that ξ_z varies on a scale << the scale of the background. This is a WKB approach. So...

Let $\epsilon \equiv \frac{1}{k_zH} << 1$. Also, $k_x \sim k_z$. Now, we must make a decision about the size of ω, by comparing it with $a/H = (\gamma g/a)^{\#} = \sqrt{\frac{\gamma g}{H}}$.

There are two choices of interest:

(i) $\omega \sim a/H$

(ii) $\omega \sim ka \sim \frac{(a/H)}{\epsilon} >> a/H$.

First, let's write $\frac{d\xi_z}{dz} = i k_z \xi_z$ with $k_z H \equiv 1/\epsilon$;  becomes

$$-k_z^2\xi_z + i k_z \xi_z \left[\frac{\omega^2 \frac{d\ln P}{dz} - k_x^2 a^2 \frac{d\ln\rho}{dz}}{\omega^2 - k_x^2 a^2}\right]$$

$$+ \frac{\xi_z}{\omega^2-k_x^2a^2}\left[\frac{\omega^4}{a^2} - 2\omega^2 k_x^2 + k_x^4 a^2 - k_x^2 g \frac{d\ln P}{dz}\left(1-\frac{1}{\gamma}\right) + g\frac{k_x^4 a^2}{\omega^2}\frac{d\ln\rho}{dz} + \frac{k_x^4 g^2}{\omega^2}\right] = 0.$$

Now (i) $\omega \sim a/H$ gives $k_x^2 a^2 \gg \omega^2$ and so the dominant terms are

$$-k_z^2\xi_z + \xi_z(-k_x^2) - \xi_z\, g\frac{k_x^2}{\omega^2}\frac{d\ln\rho}{dz} - \xi_z\frac{k_x^2 g^2}{a^2\omega^2} = 0.$$

$$\Rightarrow \quad +k^2 + \frac{g k_x^2}{\omega^2}\underbrace{\left[\frac{d\ln\rho}{dz} + \frac{1}{a^2}g\right]}_{\frac{d\ln\rho}{dz} - \frac{1}{\gamma}\frac{d\ln P}{dz} = -\frac{1}{\gamma}\frac{d\ln P\rho^{-\gamma}}{dz}} = 0.$$

$$\Rightarrow \quad \boxed{\begin{aligned}\omega^2 &= \frac{k_x^2}{k^2}\frac{g}{\gamma}\frac{d\ln P\rho^{-\gamma}}{dz} \\ &= -\frac{k_x^2}{k^2}\frac{1}{\gamma\rho}\frac{dP}{dz}\frac{d\ln P\rho^{-\gamma}}{dz} \equiv +\frac{k_x^2}{k^2}N^2\end{aligned}}$$

where N^2 is the square of the Brunt-Väisälä frequency.

If $N^2 > 0$, these are called <u>internal waves</u> or g-modes. Note that different wavenumbers have different velocities (i.e., dispersion) and that ω depends on the direction of $\vec{k}$:

$$\frac{\partial \omega}{\partial \vec{k}} = \frac{\omega}{k^2} \frac{k_z}{k_x} \left(k_z \hat{x} - k_x \hat{z} \right), \quad \text{so that } \vec{k} \cdot \frac{\partial \omega}{\partial \vec{k}} = 0.$$

We'll return to the physical cause of these waves later, after the Boussinesq approx. is introduced, ~~by~~ but, for now, note that $N^2 < 0$ (i.e., upwardly decreasing entropy) gives instability. Go boil some water and think about it.

(ii) $\omega \sim ka \gg a/H$. This gives

$$-k_z^2 + \frac{1}{\omega^2 - k_x^2 a^2} \left[\underbrace{\frac{\omega^4}{a^2} - 2\omega^2 k_x^2 + k_x^4 a^2}_{(\omega^2 - k_x^2 a^2)^2 / a^2} \right] = 0.$$

$$\Rightarrow \boxed{\omega^2 = (k_x^2 + k_z^2) a^2} \quad \text{Sound waves!}$$

These are the things we want to keep out of our analysis. So, let's ask ourselves, is there something we could have done at the beginning to get internal waves without having to carry around all these annoying sound waves? Yes.

Return to $\ast$ with $\omega \sim \epsilon k a$:

$$\frac{\delta P}{P} \simeq \frac{\omega^2}{k_x^2 a^2}\, i k_z \xi_z \gamma \quad \text{if } k_z \neq 0$$

$$\simeq \underbrace{\frac{\omega^2}{k_x^2 a^2}}_{\sim \epsilon^2 \ll 1} \frac{d\ln P}{dz}\, \xi_z \quad \text{otherwise}$$

So, perhaps we should have dropped perturbations to the gas pressure at some point. Where? In the momentum eqn? Hmm... careful now. Consider (b) with

$$\frac{\delta P}{P} \simeq \frac{\omega^2}{k_x^2 a^2}\, i k_z \xi_z \gamma \Rightarrow \xi_x = -\xi_z \frac{k_z}{k_x}, \text{ or } \vec{k}\cdot\vec{\xi} = 0.$$

looks like pressure fluctuations are enforcing (near) incompressibility. Best not drop them! And (c)?

$$-\omega^2 \xi_z = -\frac{i k_z}{\rho} P \frac{\omega^2}{k_x^2 a^2}\, i k_z \xi_z \gamma - \frac{\delta\rho}{\rho} g$$

same order!

Okay. So, pressure fluctuations are small, but not so small that they can be dropped from the momentum equation. What about the entropy equation?

$$(d) \to \frac{\delta P}{P} = \gamma \frac{\delta\rho}{\rho} - \xi_z \frac{d\ln P\rho^{-\gamma}}{dz}$$

$\sim \frac{\xi_z}{H} \frac{\omega^2}{k_x^2 a^2}$

or $\sim i k_z \xi_z \gamma \frac{\omega^2}{k_x^2 a^2}$, either way... it's small. So, drop δP

$\sim \frac{\xi_z}{H}$, required for our internal waves

from entropy equation! What does that leave us with?

$$\gamma \frac{\delta\rho}{\rho} \simeq \xi_z \frac{d\ln P\rho^{-\gamma}}{dz}$$

$\Rightarrow \frac{\delta\rho}{\rho} \sim \frac{\xi_z}{H}$. Ah! Look @ (a): $\frac{\delta\rho}{\rho} + i k_z \xi_z + i k_x \xi_x + \xi_z \frac{d\ln\rho}{dz} = 0.$

$\sim \frac{\xi_z}{H}$ $\qquad \sim k\xi_z$ $\qquad \sim \frac{\xi_z}{H}$

So, to leading order, we have $\vec{k}\cdot\vec{\xi} = 0$ — incompressibility!

Okay. Things are consistent, and we have

our Boussinesq approximation:

$$\frac{\delta P}{P} \sim \frac{1}{kH}\frac{\delta \rho}{\rho} \ll \frac{\delta \rho}{\rho} \sim \frac{\delta u}{a} \ll \frac{k\,\delta u}{\omega} \sim k\xi \sim (kH)\frac{\delta u}{a}$$

or, defining the Mach number $\mathcal{M}$ and taking it to be small ($\sim \epsilon$),

$$\frac{\delta u}{a} \sim \frac{\delta \rho}{\rho} \sim \frac{\delta T}{T} \sim \frac{1}{\mathcal{M}}\frac{\delta P}{P} \sim \frac{1}{kH} \sim \epsilon \ll 1$$

In practice, this means:

1) Assume (near) incompressibility ($\vec{\nabla}\cdot\vec{\delta u} = 0$)
2) Drop pressure perturbations everywhere EXCEPT the momentum equation. They are enforcing (near) incompressibility.
3) Keep density perturbations everywhere EXCEPT the continuity equation. They interact with gravity to give buoyancy.

Let's see how much easier this makes our lives...

$(1) \to \quad i k_x \delta u_x + i k_z \delta u_z = 0$

$(2) \to \quad -i\omega \delta u_x = -i k_x \dfrac{\delta P}{\rho}$

$$\frac{\delta P}{\rho} = -\omega \, \delta u_z \, \frac{k_z}{k_x^2}$$

$(3) \to \quad -i\omega \delta u_z = -i k_z \dfrac{\delta P}{\rho} - \dfrac{\delta \rho}{\rho} g$

$$\omega \, \delta u_z = -g \frac{i k_x^2}{k^2} \frac{\delta \rho}{\rho}$$

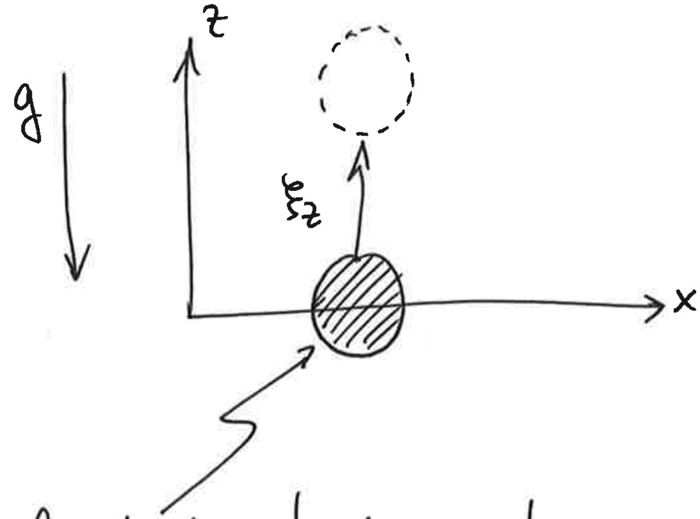
$(4) \to \quad i\omega \gamma \dfrac{\delta \rho}{\rho} + \delta u_z \dfrac{d \ln P \rho^{-\gamma}}{dz} = 0$

$$\omega^2 = + \frac{k_x^2}{k^2} \left(\frac{g}{\gamma}\right) \frac{d \ln P \rho^{-\gamma}}{dz} = \frac{k_x^2}{k^2} N^2 .$$

Much faster!!!

What we've done here is eliminated the restoring pressure forces that drive sound waves, essentially by assuming that a^2 is so large that sound waves propagate instantaneously. When the restoring force is purely external (e.g., gravity), the flow behaves as though it were incompressible (nearly). Physically, a slow moving fluid element remains in pressure balance with its surroundings.

This readjustment is what makes buoyancy waves and convection possible. Let us see that explicitly.



$P_2 < P_1$ $\quad$ $\sigma_2 > \sigma_1$

P_1 $\quad$ σ_1

where $\sigma \equiv P\rho^{-\gamma}$ is an entropy variable.

fluid element at P_1 and σ_1.

Displace the element upwards while conserving its entropy. Now it has too little entropy compared to its surroundings. With pressure balance holding, this means that it is also denser than its surroundings. It must fall back towards its equilibrium position. Overshooting, it will oscillate at N. (Mathematically, $\Delta\sigma = 0$.)

Now, consider $\sigma_2 < \sigma_1$. Now, an upwardly displaced fluid element has <u>more</u> entropy than its surroundings, and it will continue to rise $\rightarrow$ convective instability. The (Karl) Schwarzschild criterion for convective stability is $N^2 > 0$.

If you go look @ Balbus & Hawley 1991, you'll see the Boussinesq approx. in use.

Appendix A. Exact solution to  for an isothermal atmosphere.

Suppose $\frac{d\ln P}{dz} = \frac{d\ln \rho}{dz}$ ($T = \text{constant}$). Then

$$\frac{d\ln P}{dz} = -\frac{\gamma g}{a^2} = \text{const.} \Rightarrow P = P_0 \exp\left(-z/H\right) \quad \text{with } H \equiv \frac{a^2}{\gamma g}.$$

Then we have

$$\xi_z'' - \frac{\xi_z'}{H} + \left[\frac{\omega^2 - k_x^2 a^2}{a^2} + \frac{k_x^2 g}{H}\left(1 - \frac{1}{\gamma}\right)\frac{1}{\omega^2}\right]\xi_z = 0.$$

Let $\xi_z = f(z)\exp\left(\frac{z}{2H}\right)$. Then

$$\xi_z' = f' \exp\left(\frac{z}{2H}\right) + \frac{f}{2H}\exp\left(\frac{z}{2H}\right)$$
$$= f' e^{z/2H} + (\xi_z/2H)$$
$$\xi_z'' = f'' e^{z/2H} + \frac{f'}{H} e^{z/2H} + \frac{\xi_z}{(2H)^2}$$

$$f'' + \cancel{\frac{f'}{H}} + \frac{f}{(2H)^2} - \cancel{\frac{f'}{H}} - \frac{f}{2H^2} + \left[\dots\right] f = 0.$$

$$f'' + \underbrace{\left[-\frac{1}{4H^2} + \frac{\omega^2 - k_x^2 a^2}{a^2} + \frac{k_x^2 a^2}{\gamma H^2}\left(1 - \frac{1}{\gamma}\right)\frac{1}{\omega^2}\right]}_{\text{constant}} f = 0.$$

$\Rightarrow f = \exp(\pm i k_z z)$ with $k_z^2 = [\ldots]$

$$\Rightarrow -k_z^2 - \frac{1}{4H^2} + \frac{\omega^2 - k_x^2 a^2}{a^2} + \frac{k_x^2 a^2}{\omega^2 H^2}\left(\frac{\gamma-1}{\gamma^2}\right) = 0.$$

Mult. by $\omega^2 a^2$ and regroup terms:

$$\omega^4 + \omega^2\left[-k_z^2 a^2 - \frac{a^2}{4H^2} - k_x^2 a^2\right] + k_x^2 a^2 \left(\frac{a^2}{H^2}\right)\left(\frac{\gamma-1}{\gamma^2}\right) = 0.$$

Quadratic!

$$\omega^2 = \frac{k^2 a^2 + \frac{a^2}{4H^2}}{2} \pm \frac{1}{2}\left[\left(k^2 a^2 + \frac{a^2}{4H^2}\right)^2 - 4 k_x^2 a^2 \left(\frac{a^2}{H^2}\right)\left(\frac{\gamma-1}{\gamma^2}\right)\right]^{1/2}.$$

Note that $N^2 \equiv \frac{g}{\gamma}\frac{d\ln P\rho^{-\gamma}}{dz} = \left(\frac{1-\gamma}{\gamma}\right) g \frac{d\ln P}{dz} = \frac{(\gamma-1)g}{\gamma H} = \frac{a^2}{H^2}\left(\frac{\gamma-1}{\gamma^2}\right).$

So, $$\boxed{\omega^2 = \frac{k^2 a^2 + \frac{\gamma^2 N^2}{\gamma-1}}{2} \pm \frac{1}{2}\left[\left(k^2 a^2 + \frac{\gamma^2 N^2}{\gamma-1}\right)^2 - 4 k_x^2 a^2 N^2\right]^{1/2}}$$

if $k^2 H^2 \gg 1$, this becomes $\omega^2 = k^2 a^2$ for the sound waves (plus sign) and $\omega^2 = \frac{k_x^2}{k^2} N^2$ for the buoyancy waves (minus sign). The final term in the square root captures the coupling between these two waves.