M. Kunz

# Non-ideal MHD & Poorly Ionized Plasmas

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$$

what is $\vec{u}$? $= \dfrac{\sum_s m_s n_s \vec{u}_s}{\sum_s m_s n_s}$ ? $\vec{u}_i$? $\vec{u}_e$? $\vec{u}_n$?

(if $\vec{u}_i = \vec{u}_e$, where are currents?)

in ideal MHD, all of these are equivalent, because the drifts are small. For example,

$$\vec{u}_i - \vec{u}_e = (en_i)^{-1}\vec{j} = \frac{c\vec{\nabla}\times\vec{B}}{4\pi e n_i}$$

$$\Rightarrow \left|\frac{u_i - u_e}{v_A}\right| \sim \frac{c(B/\ell)}{4\pi e n_i}\frac{\sqrt{4\pi m_i n_i}}{B} \sim \frac{d_i}{\ell} \lll 1.$$

Collisions keep the neutrals co-moving with the charged species.

Let's go back to basics...

Consider a collisional plasma with neutrals, ions, and electrons.

Eqns. are

$$① \quad m_n n_n\left(\frac{\partial \vec{u}_n}{\partial t} + \vec{u}_n\cdot\vec{\nabla}\vec{u}_n\right) = -\vec{\nabla}p_n + \vec{F}_{ni} + \vec{F}_{ne}$$

$$② \quad m_i n_i\left(\frac{\partial \vec{u}_i}{\partial t} + \vec{u}_i\cdot\vec{\nabla}\vec{u}_i\right) = -\vec{\nabla}p_i + \vec{F}_{in} + \vec{F}_{ie} + q_i n_i\left(\vec{E} + \frac{\vec{u}_i\times\vec{B}}{c}\right)$$

$$③ \quad m_e n_e\left(\frac{\partial \vec{u}_e}{\partial t} + \vec{u}_e\cdot\vec{\nabla}\vec{u}_e\right) = -\vec{\nabla}p_e + \vec{F}_{en} + \vec{F}_{ei} - en_e\left(\vec{E} + \frac{\vec{u}_e\times\vec{B}}{c}\right)$$

The electric field is what ensures $q_i n_i - e n_e = 0$. Indeed, add (2) and (3):

cancel by Newton's 3rd

$$(4) \quad m_i n_i \left(\frac{D\vec{u}_i}{Dt_i}\right) + m_e n_e \left(\frac{D\vec{u}_e}{Dt_e}\right) = -\vec{\nabla}(p_i + p_e) + \vec{F}_{in} + \vec{F}_{ie} + \vec{F}_{en} + \vec{F}_{ei}$$

$$+ \underbrace{(q_i n_i - e n_e)}_{=0.}\vec{E} + \underbrace{(q_i n_i \vec{u}_i - e n_e \vec{u}_e)}_{\vec{J}}\frac{\times\vec{B}}{c}$$

Now, add (4) to (1):

$$m_n n_n \left(\frac{D\vec{u}_n}{Dt_n}\right) + m_i n_i \left(\frac{D\vec{u}_i}{Dt_i}\right) + m_e n_e \left(\frac{D\vec{u}_e}{Dt_e}\right) = -\vec{\nabla}(p_i + p_e + p_n)$$

$$+ \vec{F}_{ni} + \vec{F}_{ne} + \vec{F}_{in} + \vec{F}_{en} + \frac{\vec{J}\times\vec{B}}{c}$$

cancel by Newton's 3rd

cancel by Newton's 3rd

LHS can be written using center-of-mass velocity

$$\vec{u} = \frac{\Sigma_s m_s n_s \vec{u}_s}{\Sigma_s m_s n_s}$$

$$\rightarrow \vec{u}_s = \vec{u} + \delta\vec{u}_s$$

$$\frac{D\vec{u}_s}{Dt_s} = \frac{\partial\vec{u}_s}{\partial t} + \vec{u}_s\cdot\vec{\nabla}\vec{u}_s = \frac{\partial\vec{u}}{\partial t} + \frac{\partial}{\partial t}\delta\vec{u}_s + \vec{u}\cdot\vec{\nabla}\vec{u} + \vec{u}\cdot\vec{\nabla}\delta\vec{u}_s + \delta\vec{u}_s\cdot\vec{\nabla}\vec{u}$$

$$+ \delta\vec{u}_s\cdot\vec{\nabla}\delta\vec{u}_s$$

$$= \frac{D\vec{u}}{Dt} + \left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} + \delta\vec{u}_s\cdot\vec{\nabla}\right)\delta\vec{u}_s + \delta\vec{u}_s\cdot\vec{\nabla}\vec{u}$$

Sum over all species s:

$$\sum_s m_s n_s \frac{D\vec{u}_s}{Dt_s} = \rho \frac{D\vec{u}}{Dt} + \sum_s m_s n_s \left( \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} + \delta\vec{u}_s \cdot \vec{\nabla} \right) \delta\vec{u}_s$$

$$+ \underbrace{\sum_s m_s n_s \delta\vec{u}_s}_{=0 \text{ by def}^{\underline{n}}.} \cdot \vec{\nabla}\vec{u}$$

Note: $\vec{\nabla}\cdot\left(\sum_s m_s n_s \delta\vec{u}_s \delta\vec{u}_s\right) = \sum_s m_s n_s \delta\vec{u}_s \cdot \vec{\nabla}\delta\vec{u}_s + \sum_s m_s n_s \delta\vec{u}_s (\vec{\nabla}\cdot\delta\vec{u}_s)$

$$+ \sum_s m_s \delta\vec{u}_s \delta\vec{u}_s \cdot \vec{\nabla} n_s$$

$$\Rightarrow \sum_s m_s n_s \frac{D\vec{u}_s}{Dt_s} = \rho \frac{D\vec{u}}{Dt} + \sum_s m_s n_s \left( \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} \right) \delta\vec{u}_s$$

$$+ \vec{\nabla}\cdot\left(\sum_s m_s n_s \delta\vec{u}_s \delta\vec{u}_s\right) - \sum_s m_s n_s \delta\vec{u}_s (\vec{\nabla}\cdot\delta\vec{u}_s)$$

$$- \sum_s m_s \delta\vec{u}_s \delta\vec{u}_s \cdot \vec{\nabla} n_s$$

Also: $0 = \frac{\partial}{\partial t} \sum_s m_s n_s \delta\vec{u}_s = \sum_s m_s \frac{\partial n_s}{\partial t} \delta\vec{u}_s + \sum_s m_s n_s \frac{\partial \delta\vec{u}_s}{\partial t}$

$$= \sum_s m_s \delta\vec{u}_s \left( -\vec{\nabla}\cdot(n_s\vec{u}) - \vec{\nabla}\cdot(n_s\delta\vec{u}_s) \right) + \sum_s m_s n_s \frac{\partial \delta\vec{u}_s}{\partial t}$$

$$\Rightarrow \sum_s m_s n_s \frac{D\vec{u}_s}{Dt_s} = \rho \frac{D\vec{u}}{Dt} + \sum_s m_s \delta\vec{u}_s \vec{\nabla}\cdot(n_s\vec{u}) + \cancel{\sum_s m_s \delta\vec{u}_s \vec{\nabla}\cdot(n_s\delta\vec{u}_s)}$$

$$+ \sum_s m_s n_s \vec{u}\cdot\vec{\nabla}\delta\vec{u}_s + \vec{\nabla}\cdot\left[\sum_s m_s n_s \delta\vec{u}_s \delta\vec{u}_s\right]$$

$$\cancel{- \sum_s m_s n_s \delta\vec{u}_s (\vec{\nabla}\cdot\delta\vec{u}_s)} \cancel{- \sum_s m_s \delta\vec{u}_s \delta\vec{u}_s \cdot \vec{\nabla} n_s}$$

$$\Rightarrow \sum_s m_s n_s \frac{D\vec{u}_s}{Dt_s} = \rho \frac{D\vec{u}}{Dt} + \vec{\nabla}\cdot\Big[\underbrace{\sum_s m_s n_s \Delta\vec{u}_s \vec{u}}_{=0 \text{ by def}^{\underline{n}}}\Big] + \vec{\nabla}\cdot\Big[\sum_s m_s n_s \Delta\vec{u}_s \Delta\vec{u}_s\Big]$$

So,

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla}\cdot\left[\overleftrightarrow{I}\left(\sum_s p_s + \frac{B^2}{8\pi}\right) + \underline{\sum_s m_s n_s \Delta\vec{u}_s \Delta\vec{u}_s} - \frac{\vec{B}\vec{B}}{4\pi}\right]$$

Aside from , looks like single-fluid MHD!

So, <u>collisions</u> between neutrals and charged species is what makes the neutrals see the Lorentz force. By virtue of their large mass and the low degree of ionization in many systems, $\vec{u} \simeq \vec{u}_n$ ... so it <u>looks</u> like the neutrals are magnetized. Not true. They just need to collide often enough with the magnetized particles.

Let us return to the induction eqn:

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$$

Now, that CANNOT be the neutral velocity ... would make no sense for the magnetic flux to be frozen into a neutral fluid! Let us write

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{u}_f \times \vec{B}\right),$$

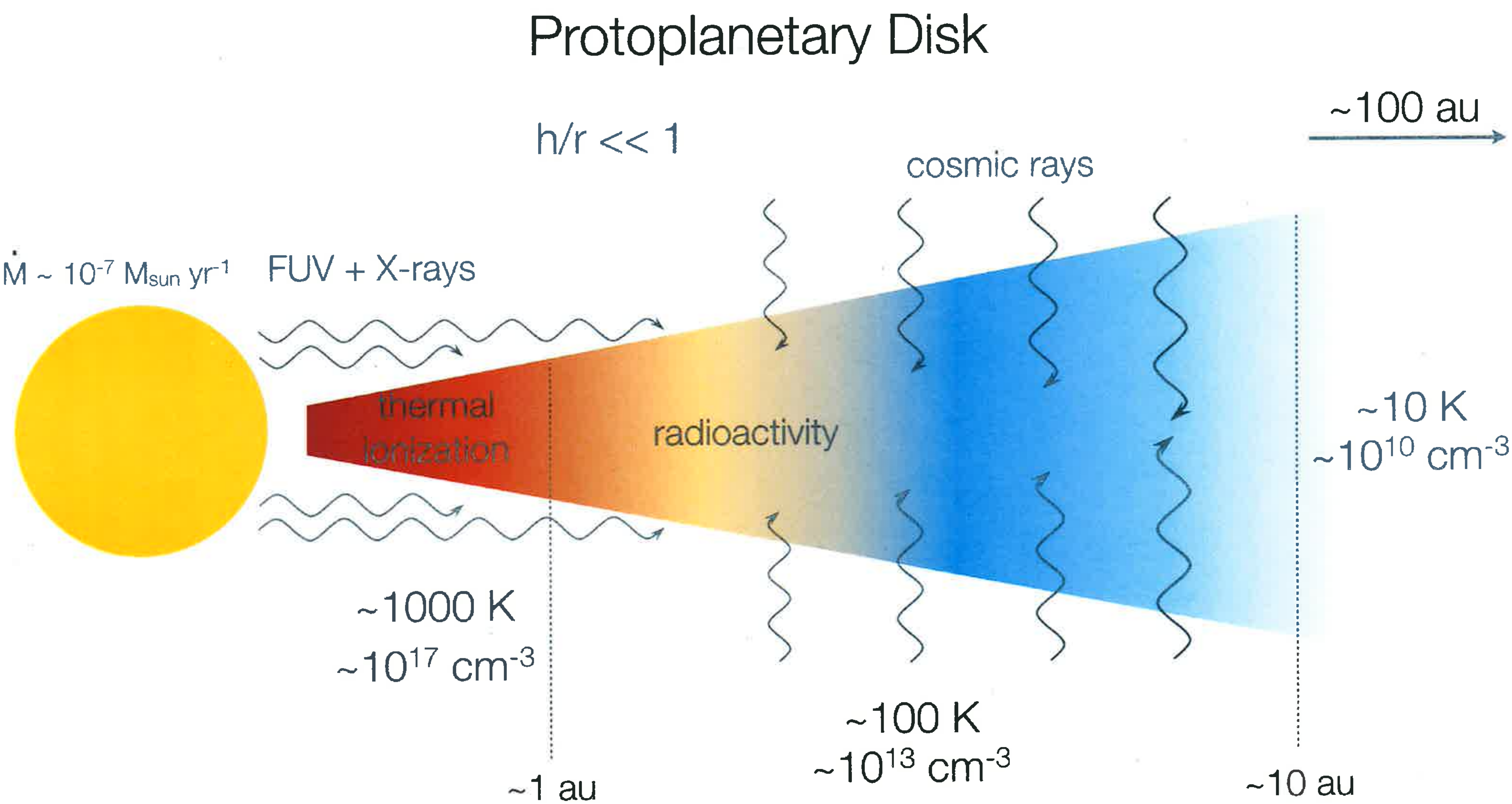
where $\vec{u}_f$ = velocity of field lines ... this must be true: field lines are frozen into themselves. Now add $\phi$:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\Bigg[\underbrace{\overset{\text{(0)}}{\underbrace{(\vec{u}_f-\vec{u}_e)}_{\substack{\text{electron-B}\\\text{drift}}}\times\vec{B}} + \overset{\text{(H)}}{\underbrace{(\vec{u}_e-\vec{u}_i)}_{\substack{\text{ion-electron}\\\text{drift}}}\times\vec{B}} + \overset{\text{(A)}}{\underbrace{(\vec{u}_i-\vec{u}_n)}_{\substack{\text{ion-neutral}\\\text{drift}}}\times\vec{B}}}_{\substack{\text{these are formally}\\\text{zero in ideal MHD!}\\\text{Let us estimate their}\\\text{sizes.}}} + \overset{\text{(I)}}{\underbrace{\vec{u}_n\times\vec{B}}_{\substack{\text{advection}\\\text{by}\\\text{neutrals}}}}\Bigg]$$

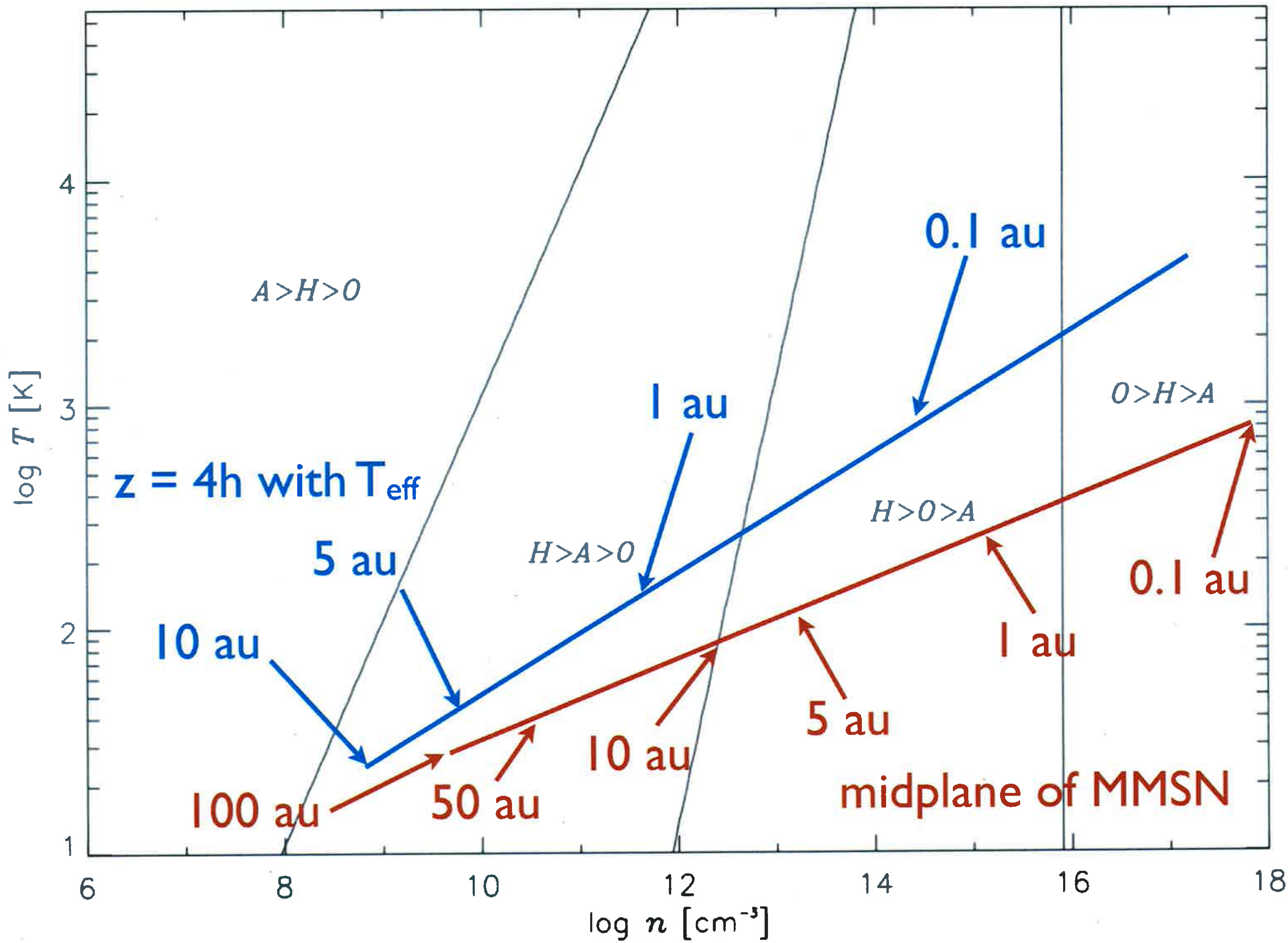
$$\frac{\text{(H)}}{\text{(I)}} \sim \left|\frac{\vec{j}/ene}{\vec{u}_n}\right| \sim \frac{c}{4\pi ene}\,\frac{B}{l}\,\frac{\sqrt{4\pi\rho}}{B}\left|\frac{v_A}{u_n}\right| \sim \underbrace{\left(\frac{d_i}{l}\right)}_{\text{small}}\underbrace{\left(\frac{\rho}{\rho_i}\right)^{1/2}}_{\substack{\sim 1 \text{ in ideal MHD,}\\\text{but could be large}}}\overbrace{\left|\frac{v_A}{u_n}\right|}^{\sim 1}$$

$$\frac{\text{(A)}}{\text{(I)}} \sim \left|\frac{\vec{F}_{ni}\tau_{ni}}{\rho_n \vec{u}_n}\right| \sim \left|\frac{\vec{j}\times\vec{B}}{c}\right|\frac{\tau_{ni}}{\rho_n v_A}\left|\frac{v_A}{u_n}\right| \sim \frac{B^2}{l}\,\frac{\tau_{ni}}{4\pi\rho_n \cancel{B}\, v_A}\left|\frac{v_A}{u_n}\right| \sim \left(\frac{v_A\tau_{ni}}{l}\right)\left(\frac{\rho}{\rho_n}\right)^{1/2}\left|\frac{v_A}{u_n}\right|$$

could be large if $\rho\sim\rho_n$ and $v_A\tau_{ni}\sim l$.




at ~1 au: $t_{\mathrm{in,coll}} \sim 3\ \mu\mathrm{s}$ $\quad t_{\mathrm{gyr,i}} \sim 40\ \mathrm{ms}$ $\quad t_{\mathrm{dyn}} \sim 1\ \mathrm{yr}$ $\quad t_{\mathrm{ni,coll}} \sim 1\ \mathrm{Myr}$




Kunz & Balbus 2004

NB: very dependent upon grain size and spatial distributions

$$\frac{(0)}{(I)} \sim \frac{1}{R_m} \equiv \frac{\eta}{V_A l}$$ — $\frac{c^2 m_e \nu_{en}}{4\pi n_e e^2}$ ← electron-neutral collision freq.

Today: Ambipolar Diffusion

Assume a poorly ionized plasma of ions (e.g., $Na^+$, $Mg^+$, $K^+$, $HCO^+$), electrons, and neutrals (e.g., $H_2$, $Na^0$, $Mg^0$) — we'll treat dust grains later — with $\vec{u}_i \simeq \vec{u}_e$ (i.e., $(d_i/l)(\rho/\rho_i)^{1/2} \ll 1$) and negligible Ohmic dissipation.

(4) $\longrightarrow$ $$m_i n_i \frac{D\vec{u}_i}{Dt_i} + m_e n_e \frac{D\vec{u}_e}{Dt_e} = -\vec{\nabla}(p_i + p_e) + \vec{F}_{in} + \vec{F}_{en} + (\vec{j}\times\vec{B}/c)$$

where $$\vec{F}_{in} = -\vec{F}_{ni} = \frac{\rho_n}{\tau_{ni}}(\vec{u}_n - \vec{u}_i)$$

$$\vec{F}_{en} = -\vec{F}_{ne} = \frac{\rho_n}{\tau_{ne}}(\vec{u}_n - \vec{u}_e)$$

and $$\tau_{ns} = a_{He-s}\,\tau_{H_2 s} = a_{He-s}\frac{m_s + m_{H_2}}{\rho_s \langle \sigma w \rangle_{sH_2}} \qquad s = i, e$$

$\langle \sigma w \rangle_{sH_2}$ ← mean collisional rate

$a_{He-s}$:

= 1.27 (s=i)
= 1.21 (s=e)
(= 1.09 (s=g))

factor by which presence of He lengthens slowing-down time relative to value it would have if only $H_2$-s collisions were considered.

$\langle \sigma w \rangle_{iH_2} = 1.69 \times 10^{-9}\ \mathrm{cm^3\,s^{-1}}$ for $HCO^+ - H_2$

(similar for $Na^+$ and $Mg^+$)

$\langle \sigma w \rangle_{eH_2} = 1.3 \times 10^{-9}\ \mathrm{cm^3\,s^{-1}}$

$$\left( \langle \sigma w \rangle_{gH_2} = \pi a_{gr}^2 \left( \frac{8 k_B T}{\pi m_{H_2}} \right)^{1/2} \text{ for } |v_n - v_g| < C_n \right)$$

radius of grain

Note: $\dfrac{\tau_{ne}}{\tau_{ni}} \simeq \dfrac{m_e + m_{H_2}}{m_e n_e} \dfrac{m_i n_i}{m_i + m_{H_2}} \sim \dfrac{m_i}{m_e} \gg 1.$

$\Rightarrow \vec{F}_{in} + \vec{F}_{en} \simeq \vec{F}_{in}$ (electrons are insignificant, relative to ions, in transmitting magn. forces to the neutrals)

④ becomes: $m_i n_i \dfrac{D\vec{u}_i}{Dt_i} = \underbrace{-\vec{\nabla}(p_i + p_e)}_{?} + \vec{F}_{in} + \dfrac{\vec{j} \times \vec{B}}{c}$

$$\frac{n_i T_i}{l} \text{ vs. } \frac{B^2}{4\pi l}$$

small due to small inertia

$\Rightarrow$ terminal speed reached fast

if $\beta \sim 1$ and $\frac{n_i}{n_n} \ll 1$, pressure is negligible

$$\Rightarrow \vec{F}_{in} \simeq -\frac{\vec{j}\times\vec{B}}{c} = \frac{\rho_n}{\tau_{ni}}\left(\vec{u}_n - \vec{u}_i\right)$$

$$\Rightarrow \boxed{\vec{u}_i \simeq \vec{u}_n + \frac{\vec{j}\times\vec{B}\,\tau_{ni}}{c\rho_n}}$$

into $\boxed{\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{u}_i\times\vec{B}\right)}$

$$\boxed{= \vec{\nabla}\times\left[\vec{u}_n\times\vec{B} + \frac{(\vec{j}\times\vec{B})\times\vec{B}}{c\rho_n \nu_{ni}}\right]}$$

$\nu_{ni} \equiv \frac{1}{\tau_{ni}}$, neutral-ion collision freq.

NB: 
$$\sum_s m_s n_s \Delta\vec{u}_s \Delta\vec{u}_s \simeq m_i n_i \left|\Delta\vec{u}_i\right|^2$$
$$\simeq m_i n_i \left|\frac{\vec{j}\times\vec{B}\,\tau_{ni}}{c\rho_n}\right|^2 \sim \left(\frac{V_A^2\tau_{ni}}{\ell}\right)^2 m_i n_i$$
$$\sim \underbrace{\left(\frac{m_i n_i}{\rho_n}\right)\left|\frac{V_A\tau_{ni}}{\ell}\right|^2}_{\ll 1} \cdot \left(\frac{B^2}{4\pi}\right)$$

So, $\rho\frac{D\vec{u}}{Dt} \simeq \boxed{m_n n_n \frac{D\vec{u}_n}{Dt} = -\vec{\nabla} p_n + \frac{\vec{j}\times\vec{B}}{c}}$

Note that $\dfrac{(\vec{J}\times\vec{B})\times\vec{B}}{c\rho_n\nu_{ni}} = \dfrac{-B^2\vec{J_\perp}}{4\pi\rho_n\nu_{ni}} = -\dfrac{V_A^2}{\nu_{ni}}\vec{J_\perp}$

$\Rightarrow$ 1) only targets $\vec{J_\perp}$ (anisotropic damping!)

2) non-linear diffusion $\propto B^2$

3) better coupling ($\nu_{ni}\to\infty$) means no A.D.

Let's do linear waves:

$$\vec{B} = \vec{B_0} + \delta\vec{B} \qquad \vec{u_n} = \delta\vec{u} \qquad P = P_0 + \delta P \qquad \rho_n = \rho_0 + \delta\rho$$

uniform

with $\delta \sim e^{-i\omega t + i\vec{k}\cdot\vec{r}}$ and work to $O(\delta)$.

Then

(A) $-i\omega\,\delta\rho + \rho_0\, i\vec{k}\cdot\delta\vec{u} = 0$

(B) $-i\omega\,\delta\vec{u} = -i\vec{k}\left(\dfrac{\delta P}{\rho_0} + \dfrac{\vec{B_0}\cdot\delta\vec{B}}{4\pi\rho_0}\right) + \dfrac{i\vec{k}\cdot\vec{B_0}}{4\pi\rho_0}\,\delta\vec{B}$

(C) $-i\omega\,\delta\vec{B} = i\vec{k}\cdot\vec{B_0}\,\delta\vec{u} - \vec{B_0}\, i\vec{k}\cdot\delta\vec{u} + \underbrace{i\vec{k}\times\left[\dfrac{(\delta\vec{J}\times\vec{B_0})\times\vec{B_0}}{c\rho_0\nu_{ni}}\right]}$

$$i\vec{k}\times\frac{\left[\vec{B_0}\vec{B_0}\cdot\delta\vec{J} - \delta\vec{J}\,B_0^2\right]}{c\rho_0\nu_{ni}}$$

$$= \frac{-(\vec{k}\times\vec{B}_0)\,\vec{B}_0\cdot(\vec{k}\times\delta\vec{B}) + B_0^2\,\vec{k}\times(\vec{k}\times\delta\vec{B})}{4\pi\rho_0\nu_{ni}}$$

$$= \frac{(\vec{k}\times\vec{B}_0)(\vec{k}\times\vec{B}_0)\cdot\delta\vec{B} - k^2B_0^2\,\delta\vec{B}}{4\pi\rho_0\nu_{ni}}$$

$$= -\frac{k^2V_{A0}^2}{\nu_{ni}}\left[\overleftrightarrow{I} - \frac{(\vec{k}\times\vec{B}_0)(\vec{k}\times\vec{B}_0)}{k^2B_0^2}\right]\cdot\delta\vec{B}$$

So,

$$\boxed{\begin{aligned}&\left[\left(-i\omega + \frac{k^2V_{A0}^2}{\nu_{ni}}\right)\overleftrightarrow{I} - \frac{(\vec{k}\times\vec{V}_{A0})(\vec{k}\times\vec{V}_{A0})}{\nu_{ni}}\right]\cdot\delta\vec{B}\\ &\qquad = i\vec{k}\cdot\vec{B}_0\,\delta\vec{u} - \vec{B}_0\, i\vec{k}\cdot\delta\vec{u}\end{aligned}}$$

Note: if $\vec{k}\parallel\vec{B}_0$, then $\omega \to \omega + i\frac{k^2V_{A0}^2}{\nu_{ni}}$. Take $\vec{k}\cdot\delta\vec{u}=0$ (incompressible)

$$\Rightarrow \left(-i\omega + \frac{k^2V_{A}^2}{\nu_{ni}}\right)\delta\vec{B} = ikB_0\,\delta\vec{u} \qquad (\text{note: } \vec{B}_0\cdot\delta\vec{B}=0 \text{ here})$$

$$\Rightarrow \frac{-i\omega}{ikB_0}\left(-i\omega + \frac{k^2V_{A0}^2}{\nu_{ni}}\right)\delta\vec{B} = -i\vec{k}\,\frac{\delta P}{\rho_0} + \frac{ikB_0}{4\pi\rho_0}\delta\vec{B}$$

$$\Rightarrow \frac{\delta P}{\rho_0} = 0 \quad (\text{since } \vec{k}\cdot\delta\vec{B}=0)$$

So... $-\omega^2 - i\omega\frac{k^2V_{A0}^2}{\nu_{ni}} + k^2V_{A0}^2 = 0 \to \omega = -i\frac{k^2V_{A0}^2}{2\nu_{ni}} \pm \left[-\left(\frac{k^2V_{A0}^2}{2\nu_{ni}}\right)^2 + k^2V_{A0}^2\right]^{1/2}$

DAMPED ALFVEN WAVES.

Let's do the general case...

$$\vec{k}\cdot \text{B} \rightarrow -i\omega\underbrace{(\vec{k}\cdot\vec{\delta u})}_{\text{A}:\ \omega\frac{\delta\rho}{\rho_0}} = -ik^2\left(\frac{\delta P}{\rho_0}+\frac{\vec{B}_0\cdot\vec{\delta B}}{4\pi\rho_0}\right) + \phi$$

$$\Rightarrow \omega^2\frac{\delta\rho}{\rho_0} = k^2\left(\frac{\delta P}{\rho_0}+\frac{\vec{B}_0\cdot\vec{\delta B}}{4\pi\rho_0}\right) = k^2a^2\frac{\delta\rho}{\rho_0} + \frac{k^2\vec{B}_0\cdot\vec{\delta B}}{4\pi\rho_0}$$

$$\Rightarrow \frac{\delta\rho}{\rho_0} = \frac{1}{a^2}\frac{\delta P}{\rho_0} = \left(\frac{k^2}{\omega^2-k^2a^2}\right)\frac{\vec{B}_0\cdot\vec{\delta B}}{4\pi\rho_0}$$

$$\text{B} \Rightarrow -i\omega\vec{\delta u} = -i\vec{k}\left(\frac{k^2a^2}{\omega^2-k^2a^2}\right)\frac{\vec{B}_0\cdot\vec{\delta B}}{4\pi\rho_0} - i\vec{k}\,\frac{\vec{B}_0\cdot\vec{\delta B}}{4\pi\rho_0} + \frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho_0}\vec{\delta B}$$

$$= -i\vec{k}\,\frac{\vec{B}_0\cdot\vec{\delta B}}{4\pi\rho_0}\left(\frac{\omega^2}{\omega^2-k^2a^2}\right) + \frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho_0}\vec{\delta B}$$

$$\text{C}\ \text{B}\ \text{A} \Rightarrow \left[\left(-i\omega+\frac{k^2v_{A0}^2}{\nu_{ni}}\right)\overleftrightarrow{I} - \frac{(\vec{k}\times\vec{v}_{A0})(\vec{k}\times\vec{v}_{A0})}{\nu_{ni}}\right]\cdot\vec{\delta B}$$

$$= \frac{\cancel{i}\vec{k}\cdot\vec{B}_0}{-\cancel{i}\omega}\left[\frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho_0}\overleftrightarrow{I} - \frac{i\vec{k}\vec{B}_0\,\omega^2}{4\pi\rho_0(\omega^2-k^2a^2)}\right]\cdot\vec{\delta B}$$

$$- \vec{B}_0\, i\omega\left(\frac{k^2}{\omega^2-k^2a^2}\right)\frac{\vec{B}_0\cdot\vec{\delta B}}{4\pi\rho_0}$$

Multiply by $i\omega$ and regroup:

$$\underbrace{\left\{ \left[ \omega^2 + i\omega \frac{k^2 V_{A0}^2}{\nu_{ni}} - (\vec{k}\cdot\vec{V_A})^2 \right] \overleftrightarrow{I} - \frac{i\omega (\vec{k}\times\vec{V_{A0}})(\vec{k}\times\vec{V_{A0}})}{\nu_{ni}} + \frac{\omega^2}{\omega^2 - k^2 a^2} \left( \vec{k}\cdot\vec{V_{A0}}\, \vec{k}\,\vec{V_{A0}} - k^2 \vec{V_{A0}}\,\vec{V_{A0}} \right) \right\}}_{\equiv \overleftrightarrow{M}} \cdot \vec{\delta B} = 0.$$

$\det \overleftrightarrow{M} = 0$ gives dispersion relation.

Algebra eased by a good coordinate system:

$$\vec{B}_0 = B_0 \hat{z}$$

$$\vec{k} = k_{\parallel}\hat{z} + k_{\perp}\hat{x}$$

$$\vec{\delta B} = \delta B_{\parallel}\hat{z} + \delta B_x \hat{x} + \delta B_y \hat{y}$$

$$\vec{k}\times\vec{V_{A0}} = -\hat{y}\, k_{\perp} V_{A0} \qquad \delta B_{\parallel} = -\frac{1}{k_{\parallel}} \left( k_{\perp} \delta B_x \right) \text{ by } \vec{k}\cdot\vec{\delta B} = 0.$$

$$\vec{V_{A0}}\cdot\vec{\delta B} = -V_{A0}\frac{k_{\perp}}{k_{\parallel}}\delta B_x$$

Write $\tilde{\omega}^2 \equiv \omega^2 + i\omega \frac{k^2 V_{A0}^2}{\nu_{ni}} - (\vec{k}\cdot\vec{V_{A0}})^2$.

$$\begin{bmatrix} \tilde{\omega}^2 - \frac{\omega^2}{\omega^2 - k^2 a^2} V_{A0}^2 k_{\perp}^2 & 0 \\ 0 & \tilde{\omega}^2 - i\omega \frac{k_{\perp}^2 V_{A0}^2}{\nu_{ni}} \end{bmatrix} \begin{bmatrix} \delta B_x \\ \delta B_y \end{bmatrix} = 0.$$

$$\rightarrow \left(\omega^2 + i\omega \frac{k_\parallel^2 V_{A0}^2}{\nu_{ni}} - k_\parallel^2 V_{A0}^2\right)\left(\omega^2 + i\omega \frac{k^2 V_{A0}^2}{\nu_{ni}} - k_\parallel^2 V_{A0}^2 - \frac{\omega^2 k_\perp^2 V_{A0}^2}{\omega^2 - k^2 a^2}\right) = 0.$$

Alfvén wave damped by ion-neutral friction

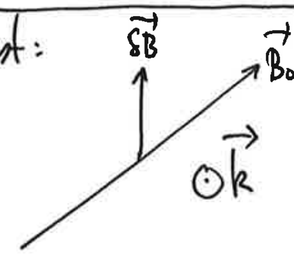
magnetosonic waves damped

$$\omega = -i\frac{k_\parallel^2 V_{A0}^2}{2\nu_{ni}} \pm \left[-\left(\frac{k_\parallel^2 V_{A0}^2}{2\nu_{ni}}\right)^2 + k_\parallel^2 V_{A0}^2\right]^{1/2}$$

$$\approx -i\frac{k_\parallel^2 V_{A0}^2}{2\nu_{ni}} \pm k_\parallel V_{A0}\sqrt{1 - \frac{k_\parallel^2 V_{A0}^2}{4\nu_{ni}^2}}$$

Damping rate

Test:




what is left after A.D.?
what is left after O.D.?

$$(\omega^2 - k^2a^2)\left(\omega^2 + i\omega\frac{k^2V_{A0}^2}{\nu_{ni}} - k_\parallel^2 V_{A0}^2\right) - \omega^2 k_\perp^2 V_{A0}^2 = 0.$$

$$\omega^4 + \omega^2\left[-k_\parallel^2 V_{A0}^2 - k^2a^2 - k_\perp^2 V_{A0}^2\right] + k^2a^2k_\parallel^2 V_{A0}^2 = -i\omega\frac{k^2V_{A0}^2}{\nu_{ni}}(\omega^2 - k^2a^2)$$

$$\underbrace{\omega^4 - \omega^2 k^2(a^2 + V_{A0}^2) + k^2a^2k_\parallel^2 V_{A0}^2}_{\text{slow \& fast}} = \underbrace{-i\omega\frac{k^2V_{A0}^2}{\nu_{ni}}(\omega^2 - k^2a^2)}_{\text{damping}}$$

NB: magnetosonic modes damped differently than Alfvén waves!

$$\lambda_{AD} = \frac{2\pi}{k} \quad \text{where} \quad k v_A \sim \frac{k^2 v_A^2}{2\nu_{ni}} \rightarrow k \sim \frac{2\nu_{ni}}{v_A}$$

$$= \pi v_A \tau_{ni} = \pi \frac{B}{\sqrt{4\pi\rho_n}} \, 1.23 \, \frac{m_i + m_{H_2}}{m_i n_i \langle\sigma w\rangle_{iH_2}}$$

$$\left( \begin{array}{l} m_i = 29\, m_H \\ \rho_n = 2.33\, m_H n_n = 2.33\, m_H (1.2)\, n_{H_2} \end{array} \right)$$

$$= \left(\frac{B}{10\,\mu G}\right)\left(\frac{n_{H_2}}{10^3\,cm^{-3}}\right)^{-1/2} \frac{3.27\times 10^{-5}}{n_i}\ pc \quad w/\ \langle\sigma w\rangle_{iH_2} = 1.69\times 10^{-9}\ \frac{cm^3}{s}$$

Use $\zeta_{cr} n_{H_2} = \alpha_{dr} n_i n_e = \alpha_{dr} n_i^2 \rightarrow n_i = \left(\frac{\zeta_{cr} n_{H_2}}{\alpha_{dr}}\right)^{1/2}$

$$\Rightarrow \boxed{\lambda_{AD} = \left(\frac{B}{10\,\mu G}\right)\left(\frac{n_{H_2}}{10^3\,cm^{-3}}\right)^{-1} 0.23\ pc} \quad w/\ \begin{array}{l} \zeta_{cr} = 5\times 10^{-17}\ s^{-1} \\ \alpha_{dr} = 2.5\times 10^{-6}\ cm^3\, s^{-1} \end{array}$$

(Umebayashi & Nakano 1990)

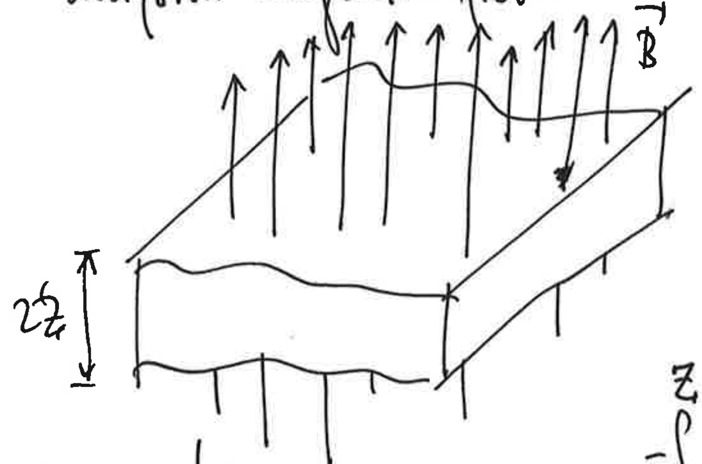
Waves below this lengthscale cannot propagate.

(Kulsrud & Pearce 1969, Appendix)

If a disturbance in the magnetic field has $\lambda < \lambda_A$, it diffuses before collisions between neutrals & ions have time to transmit to the neutrals the magnetic force associated with the disturbance. Put differently, $\tau_{AD} < \tau_A$ for $\lambda < \lambda_A$, and so neutrals are essentially unaffected by $\vec{B}$.

Another place A.D. rears its head is when self-gravity drives neutrals through field lines (and thus the ions) in molecular clouds. You'll do a problem on this for HW, but it's worth some forshadowing.

Consider a self-gravitating infinite slab of gas, threaded by a uniform magnetic field:




Surrounding pressure $= P_{ext}$

cloud sound speed $= c$

isothermal

Force balance in the vertical direction (i.e. along field lines) reads

$$-\frac{dP}{dz} = \rho_n g = 4\pi G \rho_n^2 z$$

$$-\int_0^{Z} dz\, \frac{dP}{dz} = \int_0^{Z} dz\, 4\pi G \rho_n^2 z$$

$$-P_{ext} + \rho_n c^2 = 4\pi G \rho_n^2 \frac{Z^2}{2}$$

$$= \frac{\pi G \sigma_n^2}{2} \quad \text{w/ } \sigma_n = \rho_n 2Z$$

Usually, $\frac{P_{ext}}{\rho_n c^2} \sim 0.1$, so let's drop it.

$$\rightarrow \sigma_n = \left(\frac{2\rho_n c^2}{\pi G}\right)^{1/2}$$

Mass-to-flux ratio of gas $= \frac{M}{\Phi_B} = \frac{\pi r^2 \sigma_n}{B\pi r^2} = \frac{\sigma_n}{B} = \left(\frac{2\rho_n c^2}{B^2 \pi G}\right)^{1/2}$

Now, there is a critical value of $(M/\Phi_B) = (M/\Phi_B)_{crit}$, beyond which the magnetic field cannot sustain the cloud against self-gravitationally driven fragmentation and contraction. For a slab, it is

$$\left(\frac{M}{\Phi_B}\right)_{crit} = \frac{1}{2\pi\sqrt{G}} \quad \text{(Nakano \& Nakamura 1978)}$$

( For a more realistic configuration, it is

$$\left(\frac{M}{\Phi_B}\right)_{crit} \simeq \frac{1}{\sqrt{63G}} \quad \text{(Mouschovias \& Spitzer 1976)}$$

(technically, at the center of the cloud it's $\frac{3}{2}$ times this) )

So,
$$\frac{M/\Phi_B}{(M/\Phi_B)_{crit}} = \left(\frac{4\rho_n c^2 \, 4\pi^2 G}{B^2 \, 2\pi G}\right)^{1/2} = \left(\frac{8\rho_n c^2 \pi}{B^2}\right)^{1/2}$$

Let's plug in some numbers...

$$\left.\begin{aligned} n_{H_2} &= 10^3\ \text{cm}^{-3} \\ B &= 10\,\mu G \end{aligned}\right\} \quad \frac{M/\Phi_B}{(M/\Phi_B)_{crit}} = 0.65 \left(\frac{n_{H_2}}{10^3\ \text{cm}^{-3}}\right)^{1/2} \left(\frac{10\,\mu G}{B}\right)$$

Such a cloud would be magnetically supported.

In order to initiate fragmentation and contraction, flux freezing must break down $\rightarrow$ Ambipolar Diffusion.

This leads to a redistribution of (neutral) mass in the central flux tubes of the cloud, leaving the magnetic field behind (and the ions!) as the neutrals drift through. (Mouschovias 76, 77, 78, 79; review by Mouschovias & Ciolek 99)

(You'll calculate timescales for this in HW for a variety of cloud shapes. For linear stability analysis of the above geometry, see Ciolek & Basu 2006.)

This is the classic picture that naturally accounts for the inefficiency of star formation and the observation that molecular clouds, as a whole, are not collapsing.

HW: ambipolar diffusion and the MRI and in star-forming molecular clouds

Magnetic flux problem of star formation (Babcock & Cowling 1953): take $\sim 1\,M_\odot$ of ISM and compute its magnetic flux:

$$\Phi_B = \pi r^2 B = \pi B \left(\frac{3M_\odot}{4\pi\rho}\right)^{2/3} = \left(\frac{B}{5\,\mu G}\right)\left(\frac{1\,cm^{-3}}{n_H}\right)\underbrace{5.8\times10^{32}\,Mx}_{\approx 60\,\mu G\,pc^2}$$

Flux of typical Ap star (believed to avoid convective stage as protostars and so retain fossil field) is $\sim 3\times10^{26}$ Mx.

To get an expression for the non-ideal induction eqn., we kind of cheated... several assumptions were made. We can do better. Consider the (inertialess) force equation for the charged species:

$$0 = q_s n_s \left( \vec{E} + \frac{\vec{u}_s \times \vec{B}}{c} \right) + \vec{F}_{sn}$$

Here, we've dropped pressure and gravity because $\frac{n_s}{n_n} \ll 1$ is assumed. With $\vec{F}_{sn} = \frac{\rho_n}{\tau_{ns}} (\vec{u}_n - \vec{u}_s) = \frac{\rho_s}{\tau_{sn}} (\vec{u}_n - \vec{u}_s)$, this becomes

$$0 = q_s n_s \left( \vec{E} + \frac{\vec{u}_s \times \vec{B}}{c} \right) + \frac{\rho_s}{\tau_{sn}} (\vec{u}_n - \vec{u}_s)$$

We are going to use this system of equations to obtain $\vec{E}$ — a <u>generalized Ohm's law</u>. Here, $s = i, e, \underbrace{g+, g-}_{\text{charged grains}}$;

We'll only consider elastic collisions with the neutrals, since they're the dominant collisions in most of the parameter space in molecular clouds and their cores. For an inclusion of inelastic grain-grain collisions, see the Appendix of Kunz & Mouschovias (2009a).

Start by writing $\vec{w}_s \equiv \vec{u}_s - \vec{u}_n$ and $\vec{E}_n = \vec{E} + \frac{\vec{u}_n \times \vec{B}}{c}$.

$$\Rightarrow \boxed{0 = q_s n_s \left( \vec{E}_n + \frac{\vec{w}_s \times \vec{B}}{c} \right) - \frac{\rho_s}{\tau_{sn}} \vec{w}_s}$$

Now, the current density $\vec{j} = \Sigma_s q_s n_s \vec{u}_s = \Sigma_s q_s n_s \vec{w}_s$ (by quasi-neutrality)

So, if we can write $\vec{w}_s$ in terms of $\vec{E}$, we can invert and get $\vec{E} = \vec{E}(\vec{j})$ — an Ohm's law.

Must solve boxed eqn. above. Start by taking it $\times \vec{B}$:

$$0 = q_s n_s \Big[ \vec{E}_n \times \vec{B} + \underbrace{\frac{(\vec{w}_s \times \vec{B}) \times \vec{B}}{c}}_{-\frac{B^2 \vec{w}_{s\perp}}{c}} \Big] - \frac{\rho_s}{\tau_{sn}} \vec{w}_s \times \vec{B}$$

$$\Rightarrow \vec{w}_s \times \vec{B} = \frac{q_s n_s \tau_{sn}}{m_s n_s} \left[ \vec{E}_n \times \vec{B} - \frac{B^2}{c} \vec{w}_{s\perp} \right]$$

$$\Rightarrow 0 = q_s n_s \vec{E}_n + \frac{q_s^2 n_s \tau_{sn}}{m_s c} \left[ \vec{E}_n \times \vec{B} - \frac{B^2}{c} \vec{w}_{s\perp} \right] - \frac{m_s n_s}{\tau_{sn}} \vec{w}_s$$

$\parallel$ component: $\vec{w}_{s\parallel} = \frac{q_s n_s \tau_{sn}}{m_s n_s} \vec{E}_{n\parallel} \Rightarrow$

$$\boxed{\begin{aligned} \vec{j}_\parallel &= \left( \Sigma_s \frac{q_s^2 n_s \tau_{sn}}{m_s} \right) \vec{E}_{n\parallel} \\ &\equiv (\Sigma_s \sigma_s) \vec{E}_{n\parallel} \equiv \sigma_\parallel \vec{E}_{n\parallel} \end{aligned}}$$

$\perp$ component: $0 = q_s n_s \vec{E}_{n\perp} + \frac{q_s^2 \tau_{sn} n_s}{m_s c} \vec{E}_n \times \vec{B}$

$$- \vec{w}_{s\perp} \frac{m_s n_s}{\tau_{sn}} \left[ 1 + \frac{q_s^2 B^2}{n_s} \frac{n_s \tau_{sn}^2}{m_s^2 c^2} \right] \Rightarrow$$

$$\vec{w}_{s\perp} = \frac{\tau_{sn}}{m_s n_s}\left[1+\Omega_s^2\tau_{sn}^2\right]^{-1}\left[q_s n_s \vec{E}_{n\perp} + \frac{q_s^2\tau_{sn} n_s}{m_s c}\vec{E}_n\times\vec{B}\right]$$

$$\Rightarrow \vec{j}_\perp = \sum_s \frac{q_s n_s \tau_{sn}}{m_s n_s}\left[q_s n_s \vec{E}_{n\perp} + \frac{q_s^2\tau_{sn} n_s}{m_s c}\vec{E}_n\times\vec{B}\right]\Big/\left[1+\Omega_s^2\tau_{sn}^2\right]$$

$$= \underbrace{\left[\sum_s \frac{\frac{q_s^2\tau_{sn} n_s}{m_s}}{1+\Omega_s^2\tau_{sn}^2}\right]}_{\displaystyle \sum_s \frac{\sigma_s}{1+\Omega_s^2\tau_{sn}^2} \equiv \sigma_\perp}\vec{E}_{n\perp} + \underbrace{\left[\sum_s \frac{\frac{q_s^2\tau_{sn} n_s}{m_s}\left(\frac{q_s B}{m_s c}\tau_{sn}\right)}{1+\Omega_s^2\tau_{sn}^2}\right]}_{\displaystyle \sum_s \frac{\sigma_s\Omega_s\tau_{sn}}{1+\Omega_s^2\tau_{sn}^2} \equiv -\sigma_H}\vec{E}_n\times\hat{b}$$

$$\vec{j}_\perp = \sigma_\perp\vec{E}_{n\perp} - \sigma_H\vec{E}_n\times\hat{b}$$

$$\Rightarrow \vec{j} = \sigma_\parallel\vec{E}_{n\parallel} + \sigma_\perp\vec{E}_{n\perp} - \sigma_H\vec{E}_n\times\hat{b}$$

Can invert this to get $\vec{E}_n = \eta_\parallel\vec{j}_\parallel + \eta_\perp\vec{j}_\perp + \eta_H\vec{j}\times\hat{b}$

with $\eta_\parallel \equiv \frac{1}{\sigma_\parallel}$, $\eta_\perp \equiv \frac{\sigma_\perp}{\sigma_\perp^2+\sigma_H^2}$, $\eta_H \equiv \frac{\sigma_H}{\sigma_\perp^2+\sigma_H^2}$.

$$\Rightarrow \boxed{\vec{E} = -\frac{\vec{u}_n\times\vec{B}}{c} + \underbrace{\eta_\parallel\vec{j}_\parallel + \eta_\perp\vec{j}_\perp + \eta_H\vec{j}\times\hat{b}}_{\eta_{OD}\vec{j}+\eta_{AD}\vec{j}_\perp}}$$

w/ $\eta_{OD}=\eta_\parallel$  $\eta_{AD}=\eta_\perp-\eta_\parallel$.

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \Big[ \vec{u}_n \times \vec{B} - \underbrace{c\eta_{OD}\vec{j}}_{\text{Ohmic dissipation}} - \underbrace{c\eta_{AD}\vec{j}_\perp}_{\simeq \frac{B^2}{c\rho_n \nu_{ni}} \Rightarrow \eta_{AD} \simeq \frac{B^2}{c^2\rho_n \nu_{ni}}} - \underbrace{c\eta_H \vec{j}\times\hat{b}}_{\text{Hall effect (next lecture)}} \Big]$$

## Hall effect:

Let's go back to our clean ion and electron and neutrals plasma, with $n_i = n_e \ll n_n$, and assume $\vec{u}_i = \vec{u}_n$ (i.e., no ambipolar diffusion); also drop Ohmic term. Then

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \Big[ \vec{u}_n \times \vec{B} + \underbrace{(\vec{u}_e - \vec{u}_i)}_{-\frac{\vec{j}}{en_e}} \times \vec{B} \Big]$$

$$= \vec{\nabla} \times \left[ \vec{u}_n \times \vec{B} - \frac{\vec{j}\times\vec{B}}{en_e} \right]$$

Recall the size of the Hall term relative to induction: $\frac{d_i}{\ell}\left(\frac{\rho_n}{\rho_{ni}}\right)^{1/2}$

Even though $d_i/\ell \ll 1$ in molecular clouds, $\rho_n/\rho_i \sim \left(\frac{2.33\, m_H}{29\, m_H}\right)\frac{n_n}{n_i} \gtrsim 10^7$

In the deep interiors, $\frac{\rho_n}{\rho_i} \sim 10^{12}$ or so, and so the Hall term is artificially boosted. Why?

Remember what is the inertial length... disturbances with $l \lesssim d_i$ must cope with the (large) inertia of the ions. On those scales, the ions are sluggish to respond, and so the plasma ~~disturbance~~ reacts to the disturbance quite different than it would in ideal MHD. If $\vec{u}_i = \vec{u}_n$, that means that the ions and neutrals are collisionally coupled, so now you have the inertia of the ions <u>AND</u> the inertia of all the neutrals that they are carrying around with them. This enhances the inertial length.

Now, if $\vec{u}_i \neq \vec{u}_e$, that means there is a current → electric field → magnetic field change. Let's investigate this via linear theory:

$$-i\omega\,\delta\vec{B} = i\vec{k}\times\left[\delta\vec{u}\times\vec{B}_0 - \frac{(i\vec{k}\times\delta\vec{B})\times\vec{B}_0\, c}{4\pi e n_e}\right]$$

$$= i\vec{k}\cdot\vec{B}_0\,\delta\vec{u} - \vec{B}_0\, i\vec{k}\cdot\delta\vec{u} + \frac{\vec{k}c}{4\pi e n_e}\times\left[\delta\vec{B}\,\vec{k}\cdot\vec{B}_0 - \vec{k}\,\vec{B}_0\cdot\delta\vec{B}\right]$$

$$= \underbrace{i\vec{k}\cdot\vec{B}_0\,\delta\vec{u} - \vec{B}_0\, i\vec{k}\cdot\delta\vec{u}}_{\text{normal stuff}} + \underbrace{\frac{c(\vec{k}\cdot\vec{B}_0)}{4\pi e n_e}(\vec{k}\times\delta\vec{B})}_{\vec{k}\times\delta\vec{B}\text{ suggests a rotation!}}$$

$$+i\omega\,\delta\vec{B} + \frac{c\,\vec{k}\cdot\vec{B}_0}{4\pi e n_e}(\vec{k}\times\delta\vec{B}) = -i\vec{k}\cdot\vec{B}_0\,\delta\vec{u} + \vec{B}_0\, i\vec{k}\cdot\delta\vec{u}$$

Note: if ions/neutrals are stationary, then

$$i\omega\,\delta\vec{B} + \frac{c\,\vec{k}\cdot\vec{B}_0}{4\pi e n_e}(\vec{k}\times\delta\vec{B}) = 0 \longrightarrow \boxed{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \delta B_x \\ \delta B_y \end{pmatrix}\frac{ckB_0}{4\pi e n_e} \quad \text{for } \vec{k}\parallel\vec{B}_0 \text{ : looks like a rotation!}}$$

$$\rightarrow\; i\omega\,\vec{k}\times\delta\vec{B} - \frac{c\,\vec{k}\cdot\vec{B}_0}{4\pi e n_e}k^2\,\delta\vec{B} = 0$$

$$\rightarrow\; i\omega\,\delta\vec{B} + \frac{c\,\vec{k}\cdot\vec{B}_0}{4\pi e n_e}\,\frac{c\,\vec{k}\cdot\vec{B}_0}{4\pi e n_e}\,\frac{k^2\,\delta\vec{B}}{i\omega} = 0.$$

$$\rightarrow\; \omega^2 = \left(\frac{c\,\vec{k}\cdot\vec{B}_0\,k}{4\pi e n_e}\right)^2 \Rightarrow \boxed{\omega = \pm\frac{c\,k\,\vec{k}\cdot\vec{B}_0}{4\pi e n_e}}$$ whistler wave

this is $\underline{\underline{\text{dispersive}}}$; different wavelengths travel @ different speeds

Can write $\boxed{\omega = \pm(k_\parallel v_A)(k d_i)\left(\frac{\rho}{\rho_i}\right)^{1/2}}$

Continuing... $$-i\omega\,\delta\vec{u} = -\frac{i\vec{k}\,\omega^2}{\omega^2-k^2a^2}\,\frac{\vec{B}_0\cdot\delta\vec{B}}{4\pi\rho_0} + \frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho_0}\,\delta\vec{B}$$ from before

$$\Longrightarrow\; i\omega\,\delta\vec{B} + \frac{c\,\vec{k}\cdot\vec{B}_0}{4\pi e n_e}\,\vec{k}\times\delta\vec{B} = \frac{-i\vec{k}\cdot\vec{B}_0}{-i\omega}\left[\frac{i\vec{k}\cdot\vec{B}_0}{4\pi\rho_0}\,\delta\vec{B} - \frac{i\vec{k}\,\omega^2}{\omega^2-k^2a^2}\,\frac{\vec{B}_0\cdot\delta\vec{B}}{4\pi\rho_0}\right]$$

$$\cancel{*(-i\omega)\left[\omega^2-(\vec{k}\cdot\vec{v}_{A0})^2\right]\delta\vec{B} - i\omega\frac{c\,\vec{k}\cdot\vec{B}_0}{4\pi e n_e}\vec{k}\times\delta\vec{B} = -\frac{\vec{k}\cdot\vec{B}_0\,\vec{k}\,\omega^2}{\omega^2-k^2a^2}\frac{\vec{B}_0\cdot\delta\vec{B}}{4\pi\rho_0}}$$

$$+\left(\frac{\vec{B}_0}{-i\omega}\right)\frac{k^2\omega^2}{\omega^2-k^2a^2}\,\frac{\vec{B}_0\cdot\delta\vec{B}}{4\pi\rho_0}$$

Multiply by $-i\omega$:

$$\Rightarrow \left[\omega^2 - (\vec{k}\cdot\vec{V}_{A0})^2\right]\vec{\delta B} - \frac{i\omega c\,\vec{k}\cdot\vec{B}_0}{4\pi e n_e}\,\vec{k}\times\vec{\delta B}$$

$$= \frac{\omega^2}{\omega^2 - k^2 a^2}\,\frac{\vec{B}_0\cdot\vec{\delta B}}{4\pi\rho_0}\left[\vec{B}_0 k^2 - \vec{k}\,\vec{k}\cdot\vec{B}_0\right]$$

$\vec{B}_0 = B_0\hat{z}$ $\quad\quad$ $\vec{k} = k_{\parallel}\hat{z} + k_{\perp}\hat{x}$ $\quad\quad$ $\vec{k}\times\vec{\delta B} = -k_{\parallel}\delta B_y\hat{x} + k_{\perp}\delta B_y\hat{z} + \frac{k^2}{k_{\parallel}}\delta B_x\hat{y}$

$\vec{B}_0\cdot\vec{\delta B} = -\frac{k_{\perp}B_0}{k_{\parallel}}\delta B_x$

$$\rightarrow \hat{x}: \left[\omega^2 - k_{\parallel}^2 V_{A0}^2 - k_{\perp}^2 V_{A0}^2\frac{\omega^2}{\omega^2 - k^2a^2}\right]\delta B_x = -i\omega\frac{c k_{\parallel}^2 B_0}{4\pi e n_e}\delta B_y$$

$$\hat{y}: \left(\omega^2 - k_{\parallel}^2 V_{A0}^2\right)\delta B_y - \frac{i\omega c k_{\parallel}B_0}{4\pi e n_e}\frac{k^2}{k_{\parallel}}\delta B_x = 0.$$

$$\rightarrow \omega^2 - k_{\parallel}^2 V_{A0}^2 - \frac{k_{\perp}^2 V_{A0}^2\,\omega^2}{\omega^2 - k^2a^2} = \frac{-\frac{i\omega c k_{\parallel}^2 B_0}{4\pi e n_e}\,\frac{i\omega c B_0 k^2}{4\pi e n_e}}{\omega^2 - k_{\parallel}^2 V_{A0}^2}$$

$$\Rightarrow \boxed{\begin{aligned}&\left[\omega^2 + \omega\frac{c k_{\parallel} k B_0}{4\pi e n_e} - k_{\parallel}^2 V_{A0}^2\right]\left[\omega^2 - \omega\frac{c k_{\parallel} k B_0}{4\pi e n_e} - k_{\parallel}^2 V_{A0}^2\right]\\ &= \left(\omega^2 - k_{\parallel}^2 V_{A0}^2\right)k_{\perp}^2 V_{A0}^2\frac{\omega^2}{\omega^2 - k^2a^2}\end{aligned}}$$

let $a^2 \to \infty$: Positive frequency solutions are

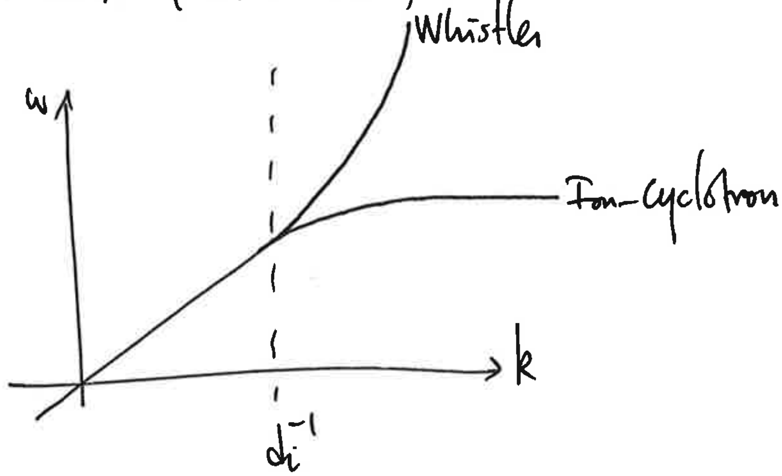
$$\omega = \mp \frac{c k_\parallel k B_0}{8\pi e n_e} + \frac{c k_\parallel k B_0}{8\pi e n_e}\sqrt{1 + \frac{16\pi e^2 n_e^2}{c^2 k^2 \rho}}$$

$$\xrightarrow{kd_i \gg 1} \underbrace{k_\parallel v_A k d_i \left(\frac{\rho}{\rho_i}\right)^{1/2}}_{\substack{\text{whistler wave} \\ \text{(right-handed)}}} \quad \text{or} \quad \underbrace{\frac{k_\parallel}{k}\frac{eB}{\mu c}\frac{n_e}{n}}_{\substack{\text{ion cyclotron wave} \\ \text{(left-handed)}}} \quad \text{w/ } \rho = \mu n$$

gets "cut off" where $\vec{\delta E}$ resonates w/ ion gyromotion

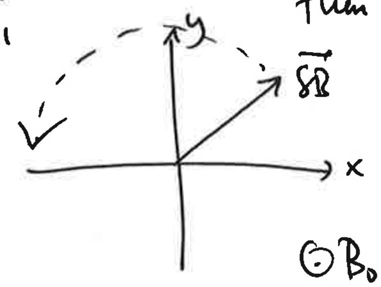
$$\xrightarrow{kd_i \ll 1} \pm k_\parallel v_A \quad \text{(Alfvén wave)}$$

With $k_\parallel = k$,




$$\frac{\delta B_y}{\delta B_x} \approx \pm \frac{i}{\omega}\frac{c k^2 B_0}{4\pi e n_e} \longrightarrow \frac{ik}{k_\parallel} \quad \text{whistler} \Rightarrow$$

if $\delta B_x \propto \cos(\omega t)$, then $\delta B_y \propto \sin(\omega t)$




rotates CCW, same direction as electrons (opp. as ions)

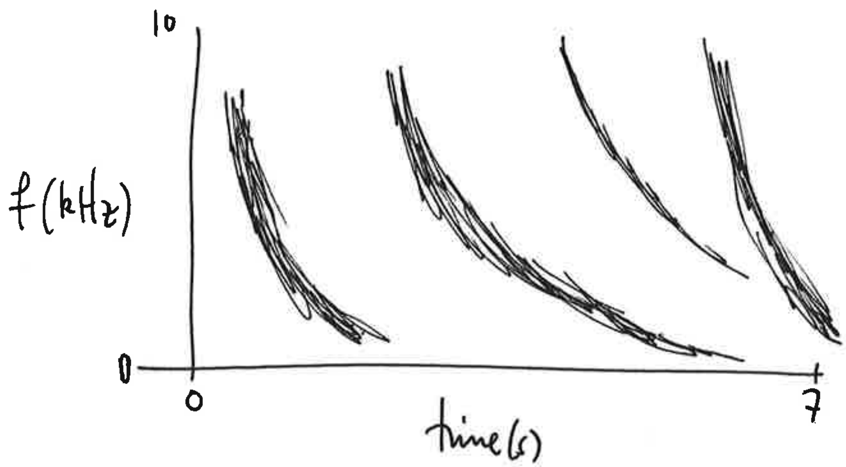
$\downarrow$

$-i(\dots)$ IC


rotates CW, opp. direction as electrons (same as ions)

NB: $\vec{\delta E}$

Whistler waves in the magnetosphere, studied for more than 100 years. Started with passive ground observations of low-frequency radio waves from the ionosphere. Some history, taken from Stenzel's 1999 Journal of Geophysical Research article.




- Preece (1894) reported that operators on the Liverpool–Hamburg telephone lines heard strange rumblings (not human produced)
- Barkhausen (1919) describes observations of "Pfeiftöne" (whistling tones) on long wire antennas and relates their occurrence to lightning and auroral activity.
- After WWII, a lot of research into whistlers started. (shouldn't be surprising ...)

Because $\omega \propto k^2 \rightarrow v \propto k$, highest $k$ (highest $\omega$) waves arrive first ... thus the "whistle" shown above.

Whistlers used to probe electron density of ionosphere.

Application to PPDs:

$T \sim 100\,K$ dominant ion is $K^+ \rightarrow m_i = 39 m_p$ (turns out not to matter)

$$l_H \equiv d_i \left(\frac{\rho}{\rho_i}\right)^{1/2} = \left(\frac{n_{H_2}}{n_i}\right)^{1/2} 2.55 \times 10^{-6}\ \text{au}$$

$$\alpha_{dr} = 5 \times 10^{-7}\ \text{cm}^3\ \text{s}^{-1} \rightarrow n_i = \left(\frac{\zeta_{cr}\, n_{H_2}}{\alpha_{dr}}\right)^{1/2} = 10^{-5} (n_{H_2})^{1/2}$$

$l_H = 0.26$ au, independent of density!

Compare to MMSN disk scaleheight (Hayashi 1981)

$$H \simeq 0.03 \left(\frac{r}{\text{au}}\right)^{5/4} \text{au}$$

$$\frac{l_H}{H} \simeq 10 \left(\frac{r}{\text{au}}\right)^{-5/4} \Rightarrow l_H \sim H \ @\ r \sim 6\ \text{au}$$

(within ~1au, cosmic rays are shielded and O.D. is ~~important~~ dominant)

Dust grains complicate this estimate greatly, especially since they tend to be the dominant charge carriers around $n_{H_2} \gtrsim 10^{12}\ \text{cm}^{-3}$.

Let us return to vorticity and Kelvin's circulation thm., with $P = P(\rho)$: $\frac{\partial \vec{\omega}}{\partial t} = \nabla \times \left[ \vec{u} \times \vec{\omega} + \frac{\vec{J} \times \vec{B}}{c\rho} \right]$. Ours new induction eqn. is $\frac{\partial \vec{B}}{\partial t} = \nabla \times \left[ \vec{u} \times \vec{B} - \frac{\vec{J} \times \vec{B}}{ene} \right]$. Look familiar?

Consider the canonical momentum

$$\vec{p}_{can} = m\vec{u} + \frac{e\vec{A}}{c}\frac{n_e}{n}$$ (Assume incompressibility)

Curl: $$\vec{\nabla} \times \vec{p}_{can} = m\vec{\omega} + \frac{e\vec{B}}{c}\frac{n_e}{n} \equiv m\vec{\omega}_{can}$$

$$\frac{\partial \vec{\omega}_{can}}{\partial t} = \frac{\partial \vec{\omega}}{\partial t} + \frac{\partial \vec{B}}{\partial t}\frac{e}{mc}\frac{n_e}{n}$$

$$= \vec{\nabla} \times \left[ \vec{u} \times \vec{\omega} + \cancel{\frac{\vec{J} \times \vec{B}}{c\rho}} + \vec{u} \times \vec{B} \xleftarrow{\frac{ene}{mcn}} - \cancel{\frac{\vec{J} \times \vec{B}}{ene}\frac{e\,n_e}{mcn}} \right]$$

$$= \vec{\nabla} \times \left( \vec{u} \times \vec{\omega}_{can} \right)$$

$$\Gamma_{can} = \oint \left( \frac{\vec{p}_{can}}{m} \right) \cdot \vec{dl}$$

local increase in magnetic flux must be accompanied by a local decrease in vorticity flux and vice versa.

In Hall-MHD, ion-neutral fluid drifts relative to the field lines and, as such, has its momentum augmented by the magnetic field through which it travels. Kind of like Lenz's law.