Radiative Transfer

2. Radiative quantities

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Introduction

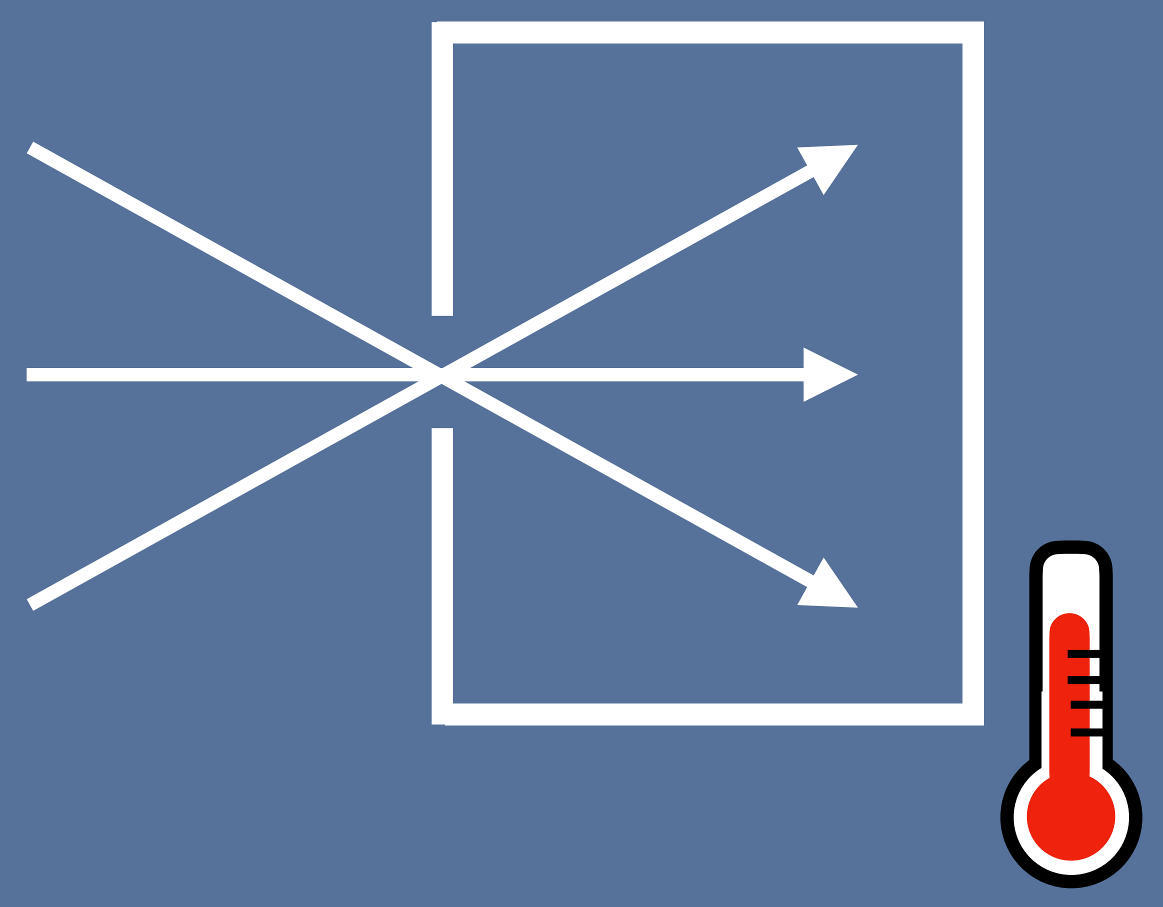
- In this chapter, we will define radiative quantities needed to describe radiation and radiative transfer
- Approximation of radiative transfer: photons move in a straight line between scattering events
- Each photon has a direction of propagation, an energy, and a polarisation
- Radiation can be seen as a photon ensemble moving through space
- We are not interested in individual photons, but rather an amount of radiative energy in a given volume.
- What we need is a quantity that contains as much information as possible

1. Luminosity

- Total energy emitted by a source per unit time (e.g. a star)
- It is a characteristic feature of the emitting object, not of the observer
- Intrinsic quantities give us information about the physics, but it is not the ones that we measure
- Unit: $erg \cdot s^{-1}$ in cgs // solar luminosity $L_\odot$
- $1\ L_\odot = 3.83\ 10^{33}\ erg \cdot s^{-1}$
- Bolometric luminosity: integrated over the whole frequency spectrum
- Monochromatic luminosity: luminosity per wavelength or frequency unit
  - L_ν with $L = \int_0^\infty L_\nu d\nu$
  - Unit: $erg \cdot s^{-1} \cdot Hz^{-1}$

2. Flux

- Thought experiment
- Cavity with black internal walls: all incoming photons are absorbed
- Insulated from the outside
- Light can penetrate through a small opening
- The heat capacity of the cavity is known
- The thermal energy coming out of the opening can be neglected
- The temperature increase as a function of time measures the total amount of energy that penetrates into the cavity per unit time
- The measured quantity should not depend on the experimental setup: the energy measured is divided by the area of the opening

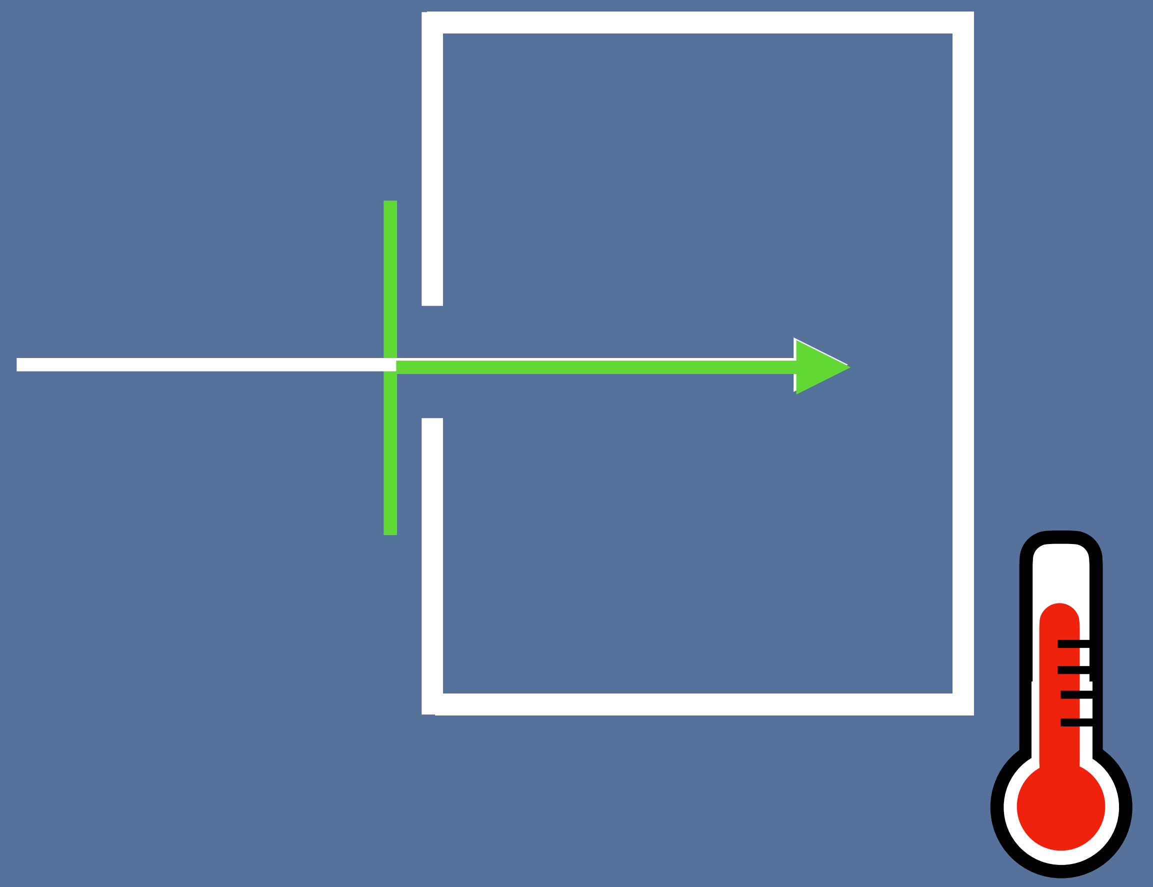


2. Flux

- We measure the flux: the energy per unit time and per unit area
- Unit: $erg \cdot s^{-1} \cdot cm^{-2}$ in cgs // $W \cdot m^{-2}$ in MKSA
- The flux is a vector quantity: the measured flux depends on the angle between the radiation source and the normal to the opening
- The previous setup measures the component F of the flux vector $\overrightarrow{F}$ along the normal to the opening $\overrightarrow{n}$
    - $F = \overrightarrow{F} . \overrightarrow{n}$
    - with 3 measures along orthogonal directions, the flux vector can be wholly determined
- Bolometric flux: integrated over the whole frequency spectrum

2. Flux

- We now place a filter in front of the opening
- The filter only lets radiation between ν_0 and $\nu_0 + \Delta\nu$ through
- The temperature increases slower, because less energy penetrates
- As before, we do not wish that the measurement depends on the experimental setup
- We therefore divide the energy by $\Delta\nu$
- This gives the monochromatic flux F_ν



2. Flux

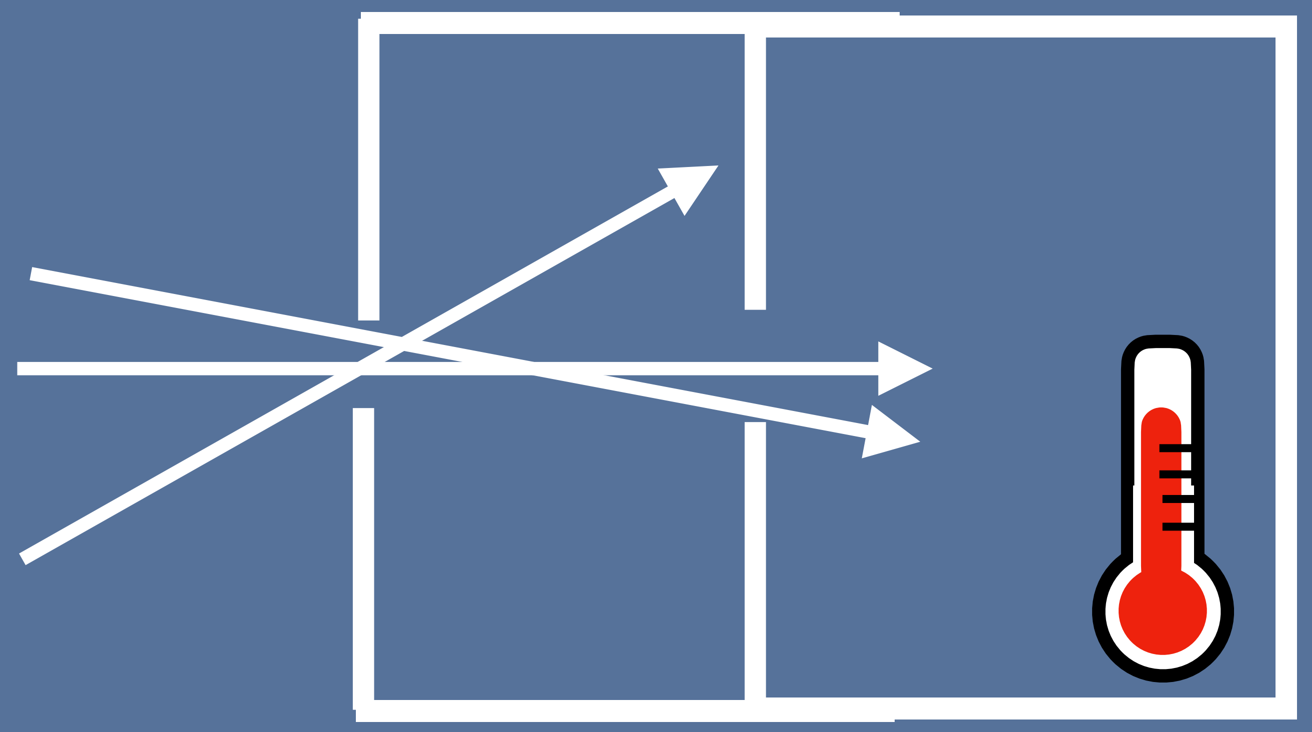
- Monochromatic flux: flux per wavelength or frequency unit — also called *flux density*
  - F_ν with $F = \int_0^\infty F_\nu d\nu$ — Unit: $erg \cdot s^{-1} \cdot cm^{-2} \cdot \ Hz^{-1}$ (cgs) // $W \cdot \ m^{-2} \cdot \ Hz^{-1}$ (MKSA)
  - F_λ with $F = \int_0^\infty F_\lambda d\lambda$ — Unit: $erg \cdot s^{-1} \cdot cm^{-2} \cdot \ cm^{-1}$ (cgs) // $W \cdot \ m^{-2} \cdot \ m^{-1}$ (MKSA)
  - It is (one of) the most widely used quantity in radiative transfer
- Other units
  - Jansky (radioastronomy) : $1 \text{ Jy} = 10^{-23} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1} = 10^{-26} \text{ W m}^{-2} \text{ Hz}^{-1}$
  - magnitudes: logarithmic scale in which the flux is compared to that of a standard star (Vega).

    Historical unit, widely used in the visible / IR, but rarely in radiative transfer

3. Intensity

3.1 Definition

- The information carried by the flux vector is still incomplete
- Another quantity is defined: the intensity I (bolometric case) and the specific intensity I_ν (monochromatic case)
- New experimental setup which includes another compartment, and an additional smaller opening
- The thermometer is placed in the rightmost cavity
- Part of the radiation passing through the first opening does not make it through the second one. We assume those photons do not contribute to the temperature increase in the setup
- Only photons passing through both openings contribute to the temperature increase
- Those are the photons that come from the direction defined by both openings



3.1 Definition

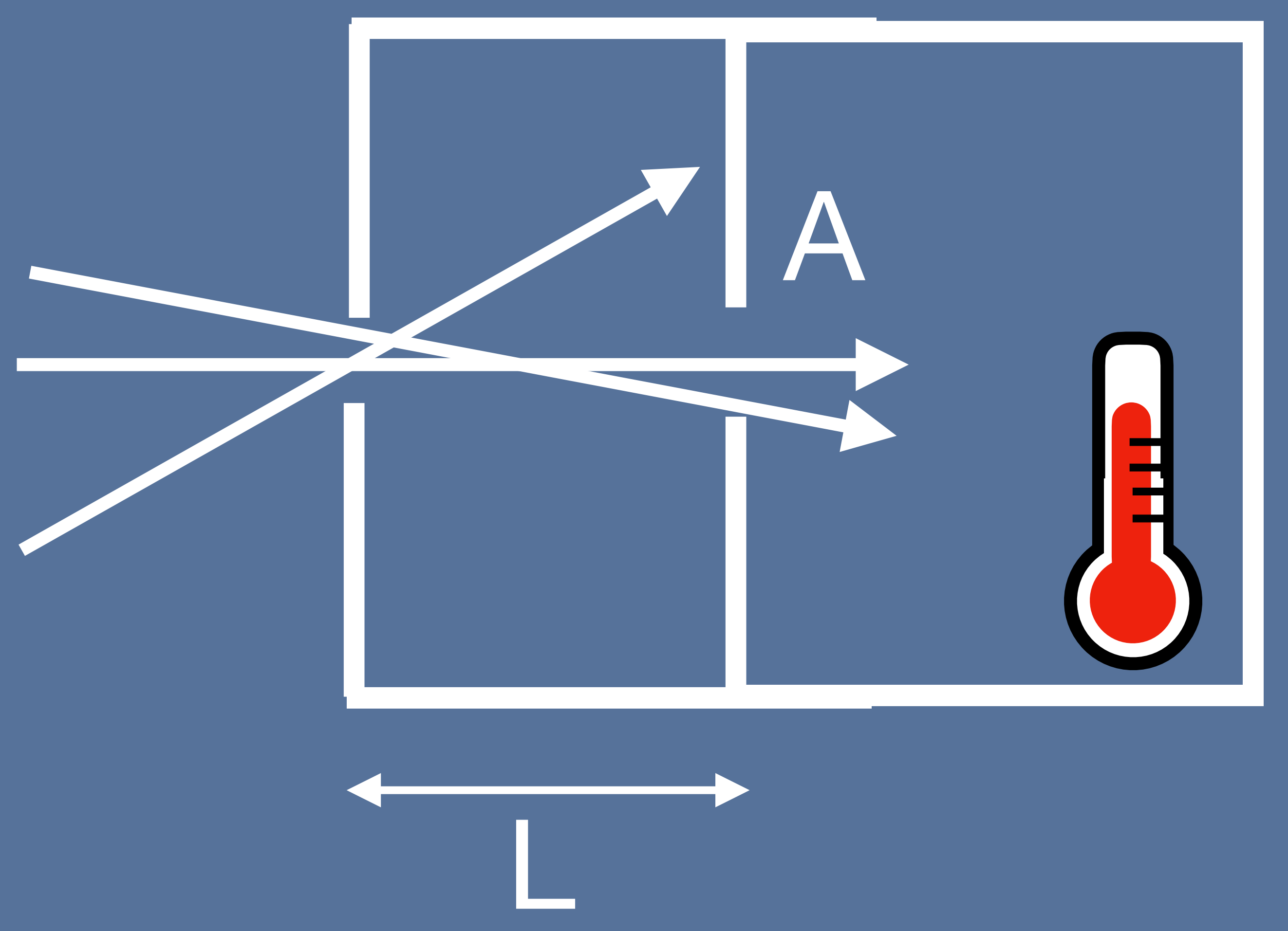
- The flux is defined as the amount of energy per unit area per unit time that passes through the first opening

- The second opening selects radiation coming from a specific direction

- If the first aperture is small with respect to the second one, the energy flux comes from a solid angle

$$\Delta\Omega = \frac{A}{L^2}$$

where A is the surface of the second aperture and L the distance between both apertures.



- If $\Delta\Omega \ll 4\pi$ the measured flux is proportional to $\Delta\Omega$, so we can divide by $\Delta\Omega$ to have a result independent from the setup

- We obtain the intensity $I = F/\Delta\Omega$

3.1 Definition

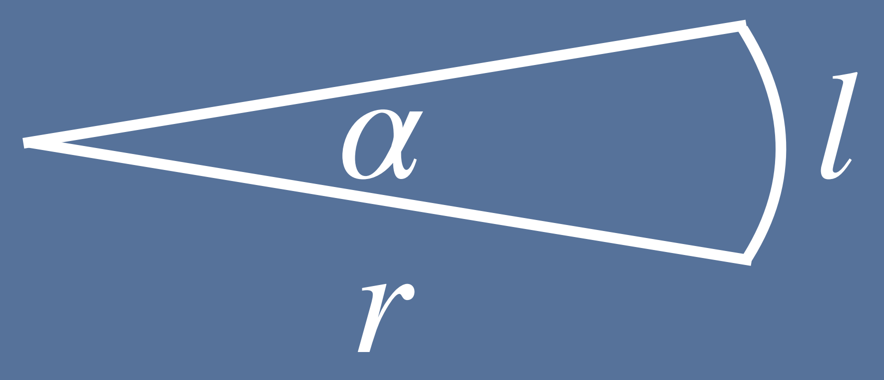
- Bolometric intensity I : amount of energy received per unit time per unit area per unit solid angle
- units: $\mathrm{erg\ s^{-1}\ cm^{-2}\ sr^{-1}}$ (cgs) $-$ $\mathrm{W\ m^{-2}\ sr^{-1}}$ (MKSA) — MJy / sr — $\mathrm{mag\,pix^{-2}}$
- As with the flux, we can add a filter of bandwidth $\Delta\nu$ and divide the amount of energy by $\Delta\nu$
- Specific intensity I_ν (monochromatic): amount of energy received per unit time per unit area per unit solid angle per unit frequency
- units: $\mathrm{erg\ s^{-1}\ cm^{-2}\ sr^{-1}\ Hz^{-1}}$ (cgs) $-$ $\mathrm{W\ m^{-2}\ sr^{-1}\ Hz^{-1}}$ (MKSA)
- Similarly, one can define I_λ. Note that: $\nu\, I_\nu = \lambda\, I_\lambda$

3.1 Definition

- The intensity is a scalar quantity (not a vector quantity) and depends on the direction
- If $\vec{n}$ is defined as a vector pointing from the first to the second aperture, then what is measured is $I(\vec{n})$ or $I_\nu(\vec{n})$.
- This is correct if the first aperture is much smaller than the second one and if the second aperture is much smaller than L.
- $I_\nu(\vec{n})$ can theoretically be measured at every point $\vec{x}$ and contains a large amount of information. In practice, one is limited by the angular resolution
- In radiative transfer the intensity is considered to have infinite resolution and is defined at all $\vec{x}$ in space, i.e. the six-dimensional function $I_\nu(\vec{x}, \vec{n})$
  - $\vec{n}$ has 3 dimensions but only two are independent
  - If we knew $I_\nu(\vec{x}, \vec{n})$ with an infinite resolution, we would have all astrophysical information! We do have surveys giving us $I_\nu(\vec{x}, \vec{n})$ for one $\vec{x}$ (the Earth) and a few ν (e.g. 2MASS in NIR)
- In radiative transfer, one has to deal with the intensity at many frequencies and $\vec{x}$ simultaneously, which is a complex problem

3.2 Solid angle and angular coordinates

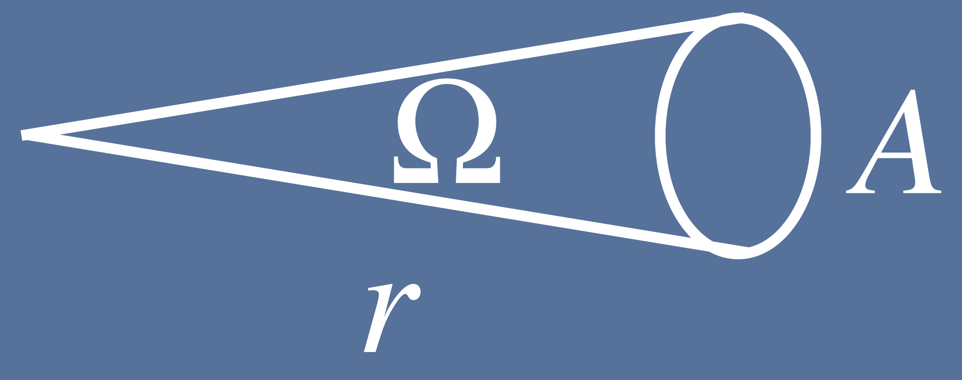
- A solid angle is the 3D equivalent of a plane angle



Plane angle (rad)

$\alpha = l/r$

is the angle that subtends the arc l



solid angle (ster or sr)

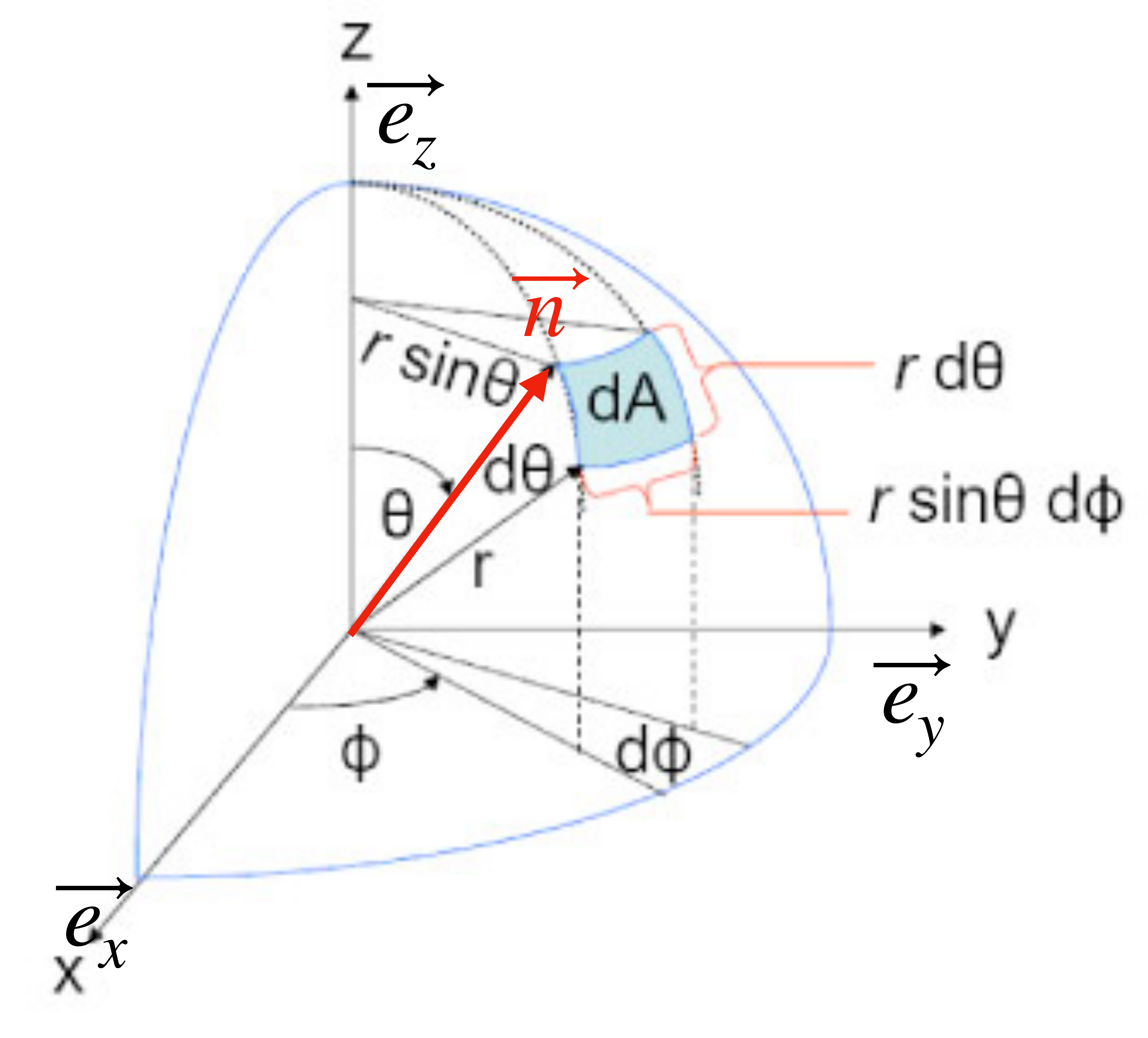
$\Omega = A/r^2$

is the solid angle that subtends the surface A (spherical surface)

- The solid angle of the whole space is the area of the unit sphere $= 4\pi$ sr
- The solid angle is a dimensionless quantity, but it does have a unit

3.2 Solid angle and angular coordinates

Solid angle in spherical coordinates



$$d\Omega = \frac{r \sin\theta d\Phi \times r d\theta}{r^2} = \sin\theta \; d\theta \; d\Phi$$

θ angle between $\overrightarrow{n}$ and $\overrightarrow{e_z}$

Φ angle between the projected vector $\overrightarrow{n'}$ in the plane xy and $\overrightarrow{e_x}$

$$\cos\theta = \overrightarrow{n} \cdot \overrightarrow{e_z} \qquad \cos\Phi = \frac{\overrightarrow{n'} \cdot \overrightarrow{e_x}}{|\overrightarrow{n'}|} \qquad \overrightarrow{n'} = \overrightarrow{n} - (\overrightarrow{n} \cdot \overrightarrow{e_z})\,\overrightarrow{e_z}$$

The components of the vector $\overrightarrow{n}$ along x, y and z are

$$n_x = \sin\theta \; \cos\Phi$$
$$n_y = \sin\theta \; \sin\Phi$$
$$n_z = \cos\theta$$

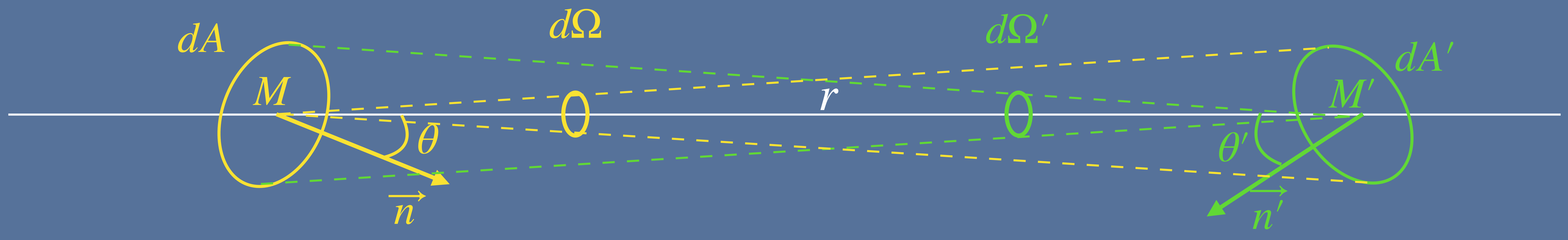
3.2 Solid angle and angular coordinates

- The intensity I_ν is therefore a function of
  - x, y, z : spatial coordinates
  - θ, Φ : angular coordinates
  - We write $I_\nu(x, y, z, \theta, \Phi)$
- In radiative transfer, it is common to define $\mu = \cos\theta$
  - We write $I_\nu(x, y, z, \mu, \Phi)$
  - $$n_x = \sqrt{1-\mu^2}\cos\Phi$$
    $$n_y = \sqrt{1-\mu^2}\sin\Phi$$
    $$n_z = \mu$$

3.3 Conservation of intensity along a beam

- The intensity measures the energy flow along a ray
- But one infinitesimal ray does not carry energy so we have to consider a collection of rays, i.e. a beam
- The beam diverges, but since the intensity is given per steradian, it is not affected by this divergence
- To show the conservation of intensity along a beam, we consider 2 infinitesimal areas dA and dA' separated by r, centred on the points M and M'
- θ is the angle between the direction MM' and $\overrightarrow{n}$, normal to dA
- θ' is the angle between the direction MM' and $\overrightarrow{n'}$, normal to dA'
- We consider the photons that pass through both surfaces dA and dA'

3.3 Conservation of intensity along a beam



- The energy carried by the photons leaving M is $dE = I_\nu \, dt \, d\nu \, dA \;\cos\theta \, d\Omega$
  - $dA \;\cos\theta$: projected (emitting) surface on MM'
  - $d\Omega$: solid angle subtended by dA' from M $$d\Omega = \frac{dA' \;\cos\theta'}{r^2}$$
- The energy carried by the photons arriving at M' is $dE' = I_\nu' \, dt \, d\nu \, dA' \;\cos\theta' \, d\Omega'$
  - $dA' \;\cos\theta'$: projected (receiving) surface on MM'
  - $d\Omega'$: solid angle subtended by dA from M' $$d\Omega' = \frac{dA \;\cos\theta}{r^2}$$

3.3 Conservation of intensity along a beam

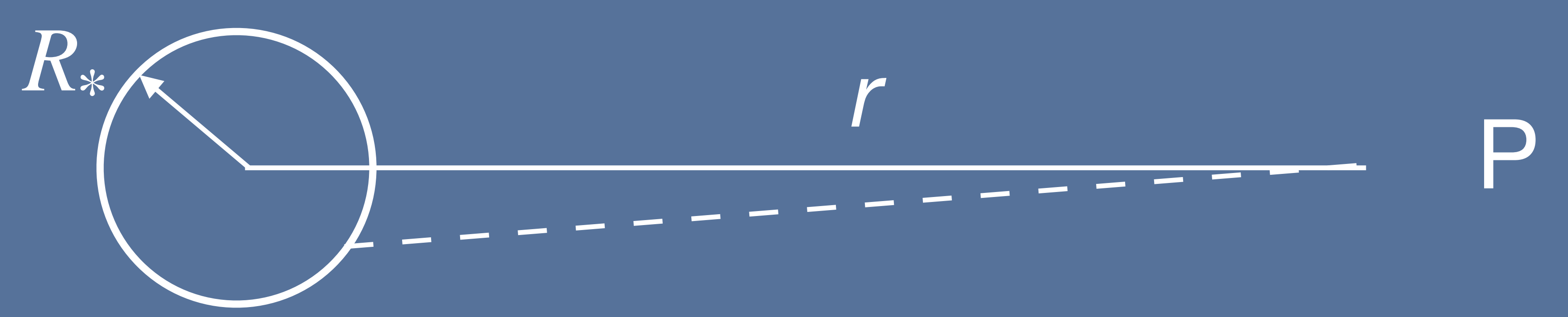
- The conservation of energy yields: $dE = dE'$
- It follows that $I_\nu = I_\nu{}'$
- The intensity is a quantity which is conserved along the direction of propagation, in the absence of emission or absorption. This property of the intensity is fundamental.
  - In vacuum the intensity is conserved.
  - This can be written vectorially $\vec{n} \cdot \vec{\nabla} I_\nu(\vec{x}, \vec{n}) = 0$ ($\vec{n} \cdot \vec{\nabla}$ is the derivative in direction $\vec{n}$)
  - The differential form is $\frac{dI_\nu(\vec{n})}{ds} = 0$, where s is the coordinate along a ray of direction $\vec{n}$. The distance along this ray is given by $\vec{x}(s) = \vec{x}_0 + s\,\vec{n}$
- This does not mean that the intensity is constant in the entire space, because it can be different for parallel rays (it depends on $\vec{x}$) and for different directions.

3.4 Flux and intensity

- Flux: energy per unit time, per unit area, per unit frequency
- Intensity: energy per unit time, per unit area, per unit frequency, per unit solid angle
- The flux is given by integrating the intensity over the directions
  - $\overrightarrow{F_\nu} = \oint I_\nu(\overrightarrow{n})\overrightarrow{n}\, d\Omega$
  - A common expression is also $F = \int I_\nu \cos\theta \, d\Omega$ which is the z-component of the flux. This is used in plane parallel or 1D geometries.
- What is the value of the flux when the intensity is isotropic?

3.4 Flux and intensity

- Contrary to the intensity, the flux is not conserved
- Calculate the flux at point P at a distance *r* from a star (assumed to be a uniform sphere radiating isotropically)

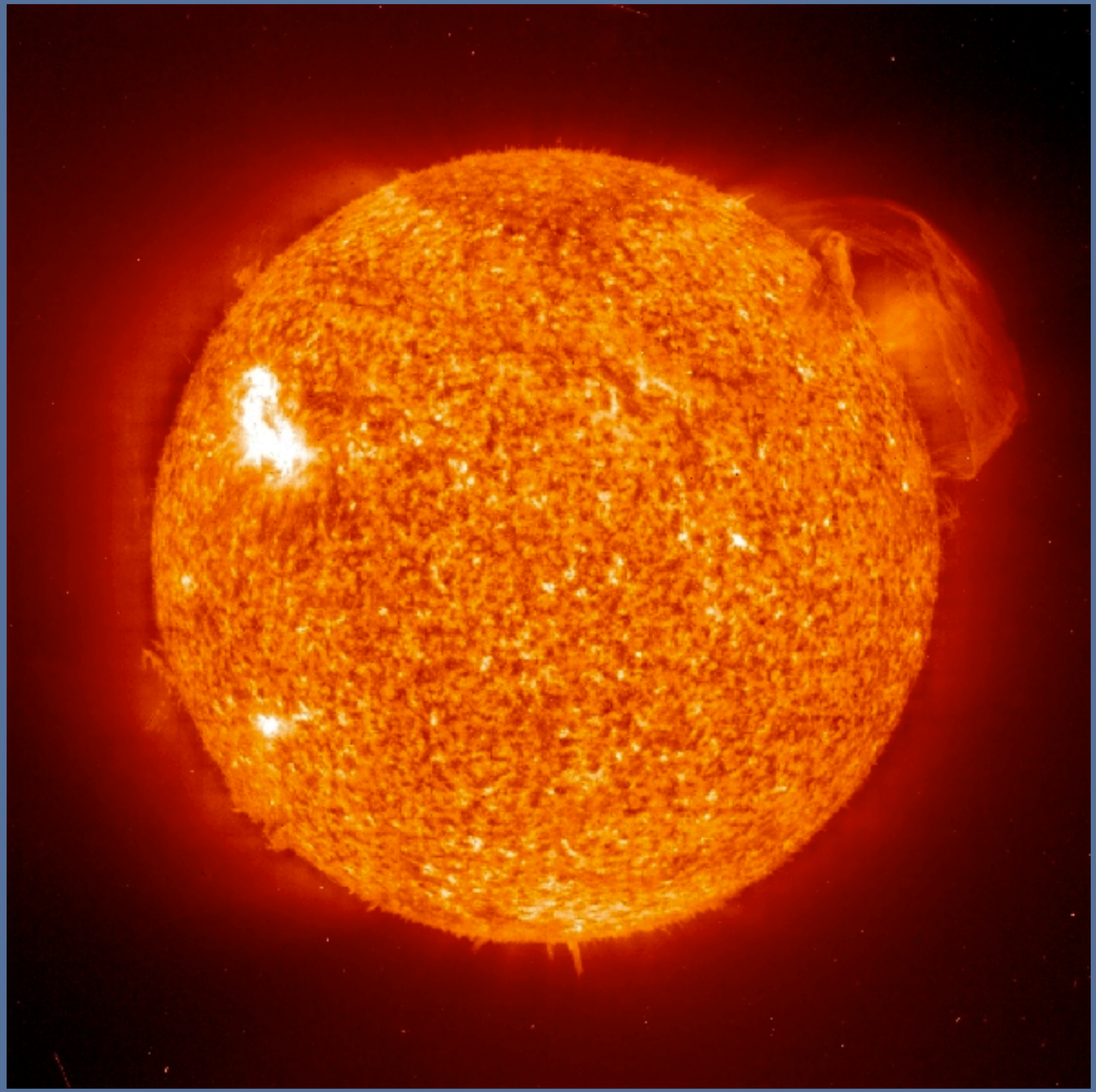


$$F_\nu(r) = \pi\, I_\nu \,\frac{R_*^2}{r^2} = F_\nu(R_*)\,\frac{R_*^2}{r^2}$$

- The flux decreases with the square of the distance
- Which quantity to use between flux and intensity
  - In physics, we like using quantities which are conserved
  - The flux does not contain information on the direction of the photons
  - The intensity is hard to work with, as it has 6 dimensions in the general case: $I_\nu(x, y, z, \theta, \Phi, t)$

3.4 Flux and intensity

Do we measure the flux of the intensity?



Extreme UV 304 Å (He II) NASA/SOHO

3.4 Flux and intensity

Do we measure the flux of the intensity?

Sirius

4. Moments of intensity

- The intensity contains much information, which makes radiative transfer hard to solve
- It is possible to develop the intensity along tensor moments
  - The complete method can be found in Thorne 1981, MNRAS, 194, 439
- In most cases, it is enough to consider the first three moments
- Zeroth order moment
  - $J_\nu = \frac{1}{4\pi} \oint I_\nu(\overrightarrow{n})\, d\Omega$, where $\overrightarrow{n}$ is the direction of the ray
  - J_ν is the mean intensity. It is the angular mean of the intensity.
  - What is J_ν for an isotropic source?

4. Moments of intensity

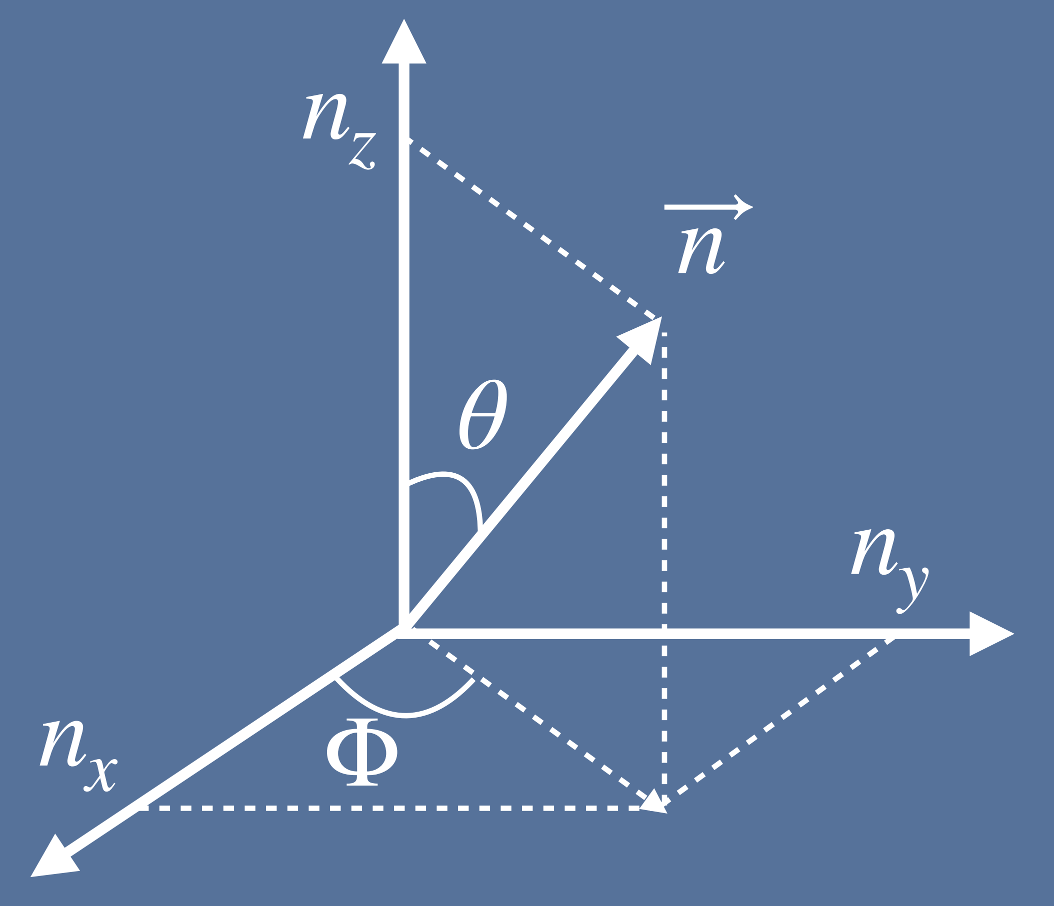
- First order moment
  - $\overrightarrow{H_\nu} = \frac{1}{4\pi} \oint I_\nu(\overrightarrow{n})\, \overrightarrow{n}\, d\Omega$
  - $\overrightarrow{H_\nu}$ is the (Eddington) flux. It is a vector quantity.
  - Its components are $H_{\nu,i} = \frac{1}{4\pi} \oint I_\nu(\overrightarrow{n})\, (\overrightarrow{e_i} \cdot \overrightarrow{n})\, d\Omega$
  - It is the flux we talked about previously, to within a factor 4π
  - $\overrightarrow{F_\nu} = \oint I_\nu(\overrightarrow{n})\, \overrightarrow{n}\, d\Omega = 4\pi\, \overrightarrow{H_\nu}$
  - Both can be used, depending on which is more convenient
  - What is $\overrightarrow{H_\nu}$ for an isotropic source?

4. Moments of intensity

- Second order moment
  - $\overline{\overline{K_\nu}} = \frac{1}{4\pi} \oint I_\nu(\overrightarrow{n})\, \overrightarrow{n}\, \overrightarrow{n}\, d\Omega$
  - $\overline{\overline{K_\nu}}$ is a symmetric tensor of rank 2, of dimensions 3 x 3.
  - The components of $\overline{\overline{K_\nu}}$ are: $\overline{\overline{K}}_{\nu,ij} = \frac{1}{4\pi} \oint I_\nu(\overrightarrow{n})\, (\overrightarrow{e_i} \cdot \overrightarrow{n})\, (\overrightarrow{e_j} \cdot \overrightarrow{n})\, d\Omega$
  - For homogeneous and isotropic radiation, $\overline{\overline{K_\nu}} = \frac{1}{3}\, \overline{\overline{\delta}}\, J_\nu$ where $\overline{\overline{\delta}}$ is the unit tensor of rank 2
  - $\overline{\overline{K_\nu}}$ is linked to the radiation pressure: $\overline{\overline{p_\nu}} = \frac{4\pi}{c}\, \overline{\overline{K_\nu}}$

4. Moments of intensity

- These moments can be written explicitely in cartesian coordinates, using $\mu = \cos\theta$
  - $J_\nu = \frac{1}{4\pi}\int_{-1}^{+1} d\mu \int_0^{2\pi} I_\nu(\mu, \Phi)\, d\Phi$
  - $H_{\nu,i} = \frac{1}{4\pi}\int_{-1}^{+1} d\mu \int_0^{2\pi} I_\nu(\mu, \Phi)\, n_i\, d\Phi$
  - $K_{\nu,i,j} = \frac{1}{4\pi}\int_{-1}^{+1} d\mu \int_0^{2\pi} I_\nu(\mu, \Phi)\, n_i\, n_j\, d\Phi$



n_i, n_j are the projections of $\vec{n}$ on the axes x, y, z

4. Moments of intensity

- For a 1D or plane parallel geometry we can reduce the problem to the z axis
  - $J_\nu = \dfrac{1}{4\pi} \oint I_\nu \, d\Omega$
  - $H_\nu = \dfrac{1}{4\pi} \oint I_\nu \, \cos\theta \, d\Omega$
  - $K_\nu = \dfrac{1}{4\pi} \oint I_\nu \, \cos^2\theta \, d\Omega$

4. Moments of intensity

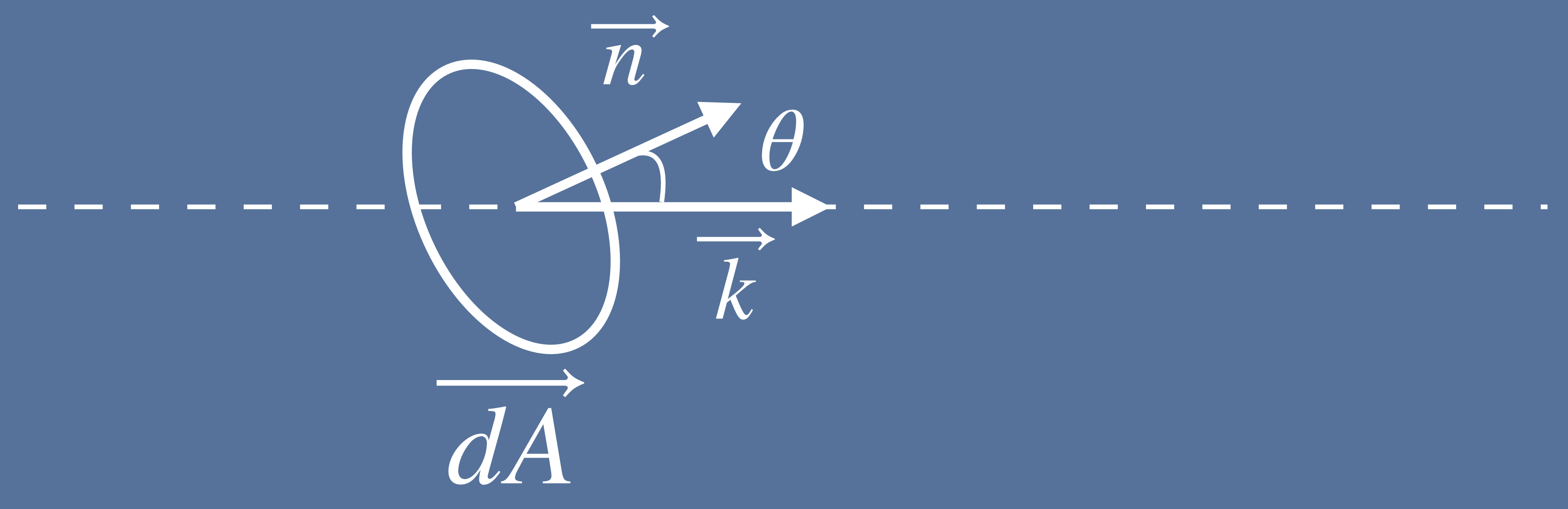
- Radiation pressure is due to the momentum carried by photons
  - For one photon of frequency ν, the momentum is $h\nu/c$: $\overrightarrow{p_p} = \dfrac{h\nu}{c}\overrightarrow{k}$
  - Momentum of a ray of intensity I_ν : $d\overrightarrow{p_\nu} = \dfrac{I_\nu}{c}(\overrightarrow{dA}\cdot\overrightarrow{k})\,dt\,d\Omega\,\overrightarrow{k}$

- The pressure applied by one ray of intensity I_ν in the direction $\overrightarrow{n}$ is:

$$dP_\nu = \frac{d\overrightarrow{p_\nu}\cdot\overrightarrow{n}}{dA\,dt} = \frac{I_\nu}{c}\cos\theta\,d\Omega\,\overrightarrow{k}\,\overrightarrow{n}$$

- For all the rays we therefore have

$$P_\nu = \frac{1}{c}\oint I_\nu \cos^2\theta\, d\Omega$$

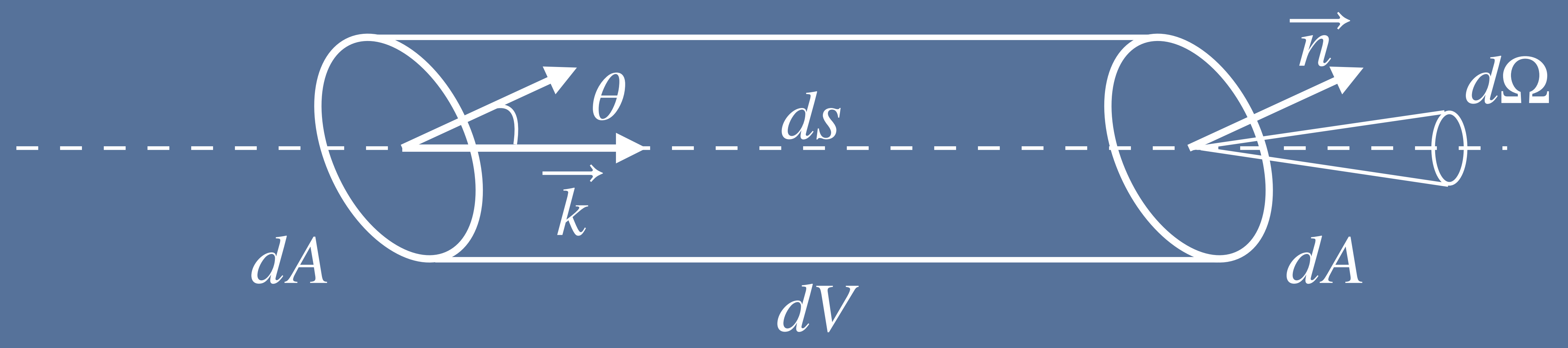


$\overrightarrow{k}$ is the direction of propagation

$\overrightarrow{dA} = dA\,\overrightarrow{n}$ is the surface of normal $\overrightarrow{n}$

4. Moments of intensity

- The energy density u_ν is the energy per unit volume



- The energy per unit frequency crossing the area dA during dt in a solid angle $d\Omega$ is: $dE_\nu = I_\nu \cos\theta \, dA \, d\Omega \, dt$

- This energy is contained in a volume $dV = \cos\theta \, dA \, ds = \cos\theta \, dA \, c \, dt$

- The integration yields: $E_\nu = \dfrac{1}{c} \displaystyle\int_{\Delta V} \int_{\Omega} I_\nu \, d\Omega \, dV$

- From the definition of $u_\nu = \dfrac{E_\nu}{\Delta V}$, we obtain $u_\nu = \dfrac{1}{c} \displaystyle\int I_\nu \, d\Omega$

- unit: $erg \, cm^{-3} \, Hz^{-1}$

4. Moments of intensity

- The energy density u_ν can be linked to the mean intensity J_ν
  - $u_\nu = \dfrac{4\pi}{c} J_\nu$
- For isotropic radiation, the radiation pressure is easily expressed as function of the energy density
  - $p_\nu = \dfrac{u_\nu}{3}$
  - $u_\nu = \dfrac{I_\nu}{c} \displaystyle\int d\Omega = \dfrac{4\pi}{c} I_\nu$
  - $p_\nu = \dfrac{I_\nu}{c} \displaystyle\int \cos^2\theta d\Omega = \dfrac{2\pi}{c} I_\nu \int_o^\pi \cos^2\theta \sin\theta d\theta = \dfrac{2\pi}{c} I_\nu \left[\dfrac{\cos^3\theta}{3}\right]_\pi^0 = \dfrac{4\pi}{3c} I_\nu$

5. Stokes parameters and Stokes vector

- Radiation can be polarised (e.g. scattering can polarise radiation)
- To describe polarisation, we need to consider the electromagnetic waves
- The EM waves are solutions to the Maxwell equations, in particular, if the propagation direction is along $\overrightarrow{e_z}$
  - $\overrightarrow{E}$ the electric field vector is perpendicular to the propagation direction
  - $\overrightarrow{B}$ the magnetic field is perpendicular to the propagation direction
  - $\overrightarrow{B} \perp \overrightarrow{E}$ and $|\overrightarrow{E}| = |\overrightarrow{B}|$
- At a given point P, the components of the electric field are
  - $E_x(P, t) = E_{x,0} \cos(\omega t - \phi_x)$
  - $E_y(P, t) = E_{y,0} \cos(\omega t - \phi_y)$
  - $\omega = 2\pi\nu$ is the angular frequency, ϕ_x and ϕ_y are the phases
  - $\Delta = \phi_y - \phi_x$ is the phase difference between both components. If $\Delta > 0$, the y component is late with respects to x

5. Stokes parameters and Stokes vector

- At a given $\overrightarrow{x}$ the EM wave can be described by
  - $E_x(\overrightarrow{x}, t) = E_{x,0} \cos(\omega t - \overrightarrow{k} \cdot \overrightarrow{x} - \phi_x)$
  - $E_y(\overrightarrow{x}, t) = E_{y,0} \cos(\omega t - \overrightarrow{k} \cdot \overrightarrow{x} - \phi_y)$
  - $\overrightarrow{k}$ is the wave vector, $|\overrightarrow{k}| = \omega/c$
- The electric field vector is $\overrightarrow{E}(\overrightarrow{x}, t) = E_x(\overrightarrow{x}, t)\,\overrightarrow{e_x} + E_y(\overrightarrow{x}, t)\,\overrightarrow{e_y}$
- The magnetic field $\overrightarrow{B}(\overrightarrow{x}, t) = \overrightarrow{e_z} \wedge \overrightarrow{E}(\overrightarrow{x}, t)$
- The mean Poynting vector (flux vector) $\overrightarrow{F} = < \overrightarrow{E} \wedge \overrightarrow{B} >$
- A perfectly coherent wave is entirely described by its propagation direction $\overrightarrow{k}$, its frequency ν, its amplitudes $E_{x,0}$ and $E_{y,0}$ and its phase difference Δ (coherent = constant phase difference)

5. Stokes parameters and Stokes vector

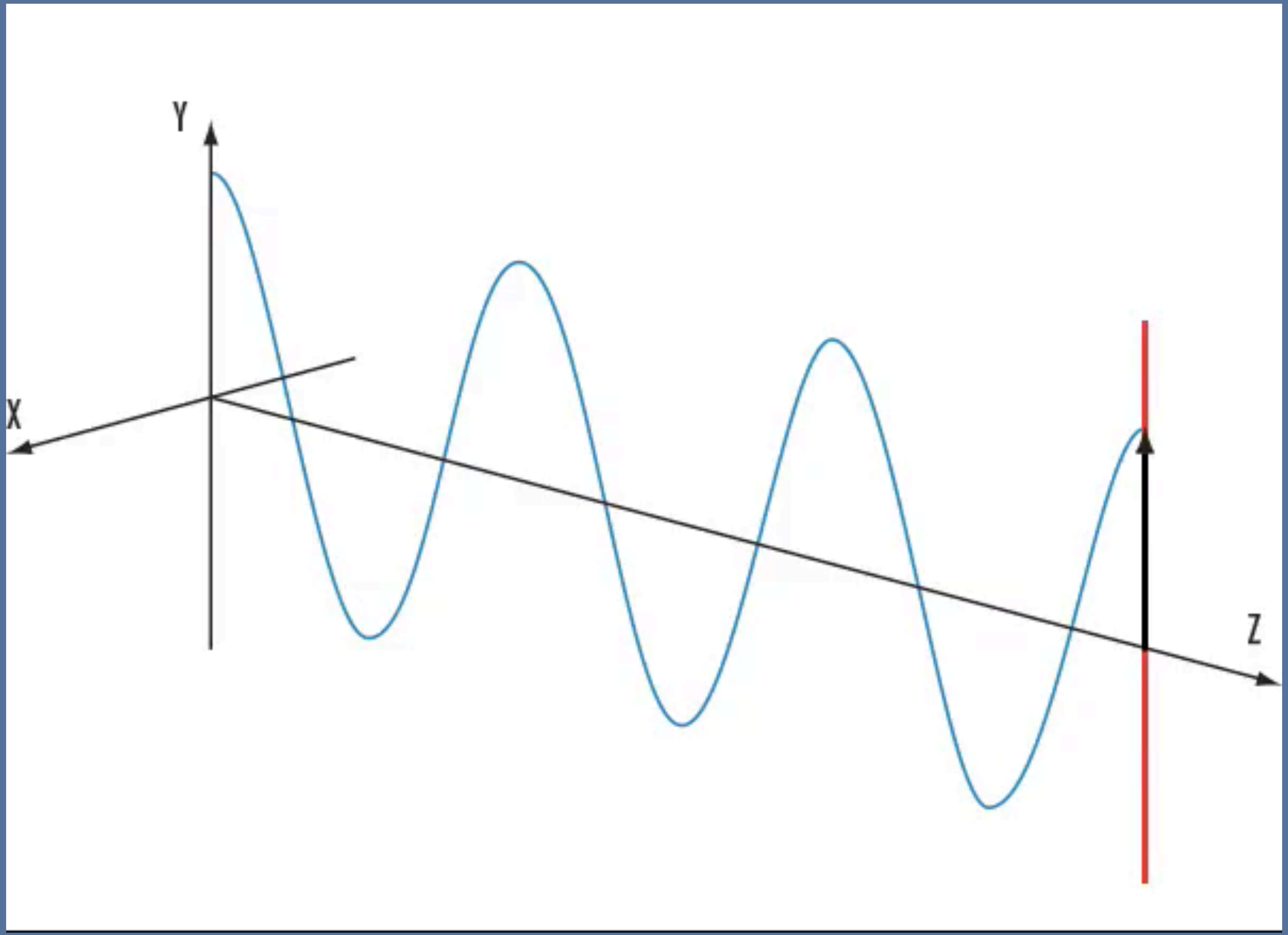
- For a coherent or a polarised wave, the radiation can be described with the Stokes vector // parameters

$$
\begin{aligned}
I &= E_{x0}^2 + E_{y0}^2 \\
Q &= E_{x0}^2 - E_{y0}^2 \\
U &= 2\,E_{x0}\,E_{y0}\,\cos\Delta \\
V &= 2\,E_{x0}\,E_{y0}\,\sin\Delta
\end{aligned}
$$

  - I is the total flux (or the intensity)
  - Q, U, V have the same dimension as I, but describe the polarisation state
  - If $Q = U = V = 0$, the radiation is not polarised
  - For completely polarised radiation $I^2 = Q^2 + U^2 + V^2$
  - In the general case $0 < Q^2 + U^2 + V^2 < I^2$

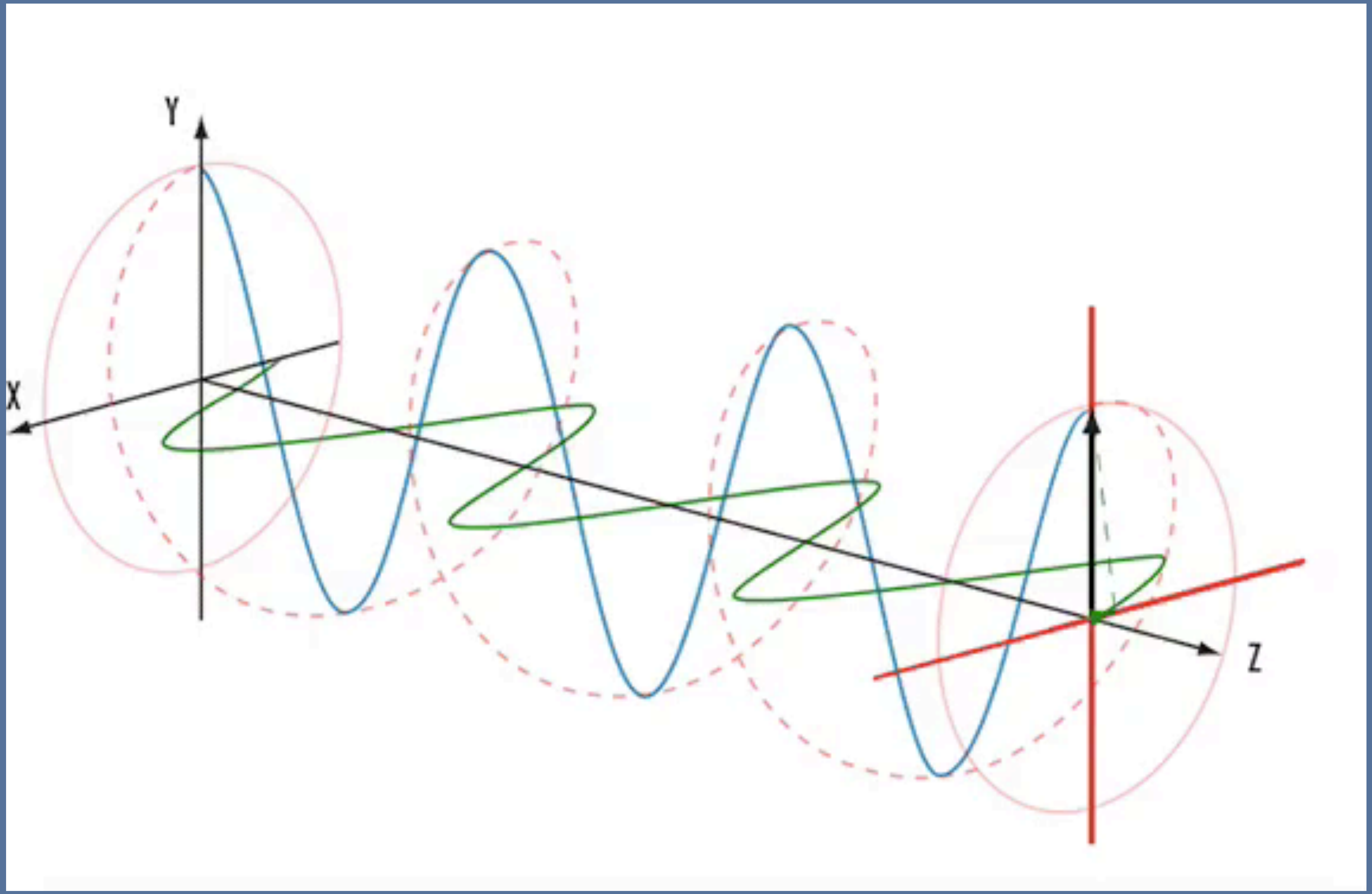
5. Stokes parameters and Stokes vector

Linearly polarised light



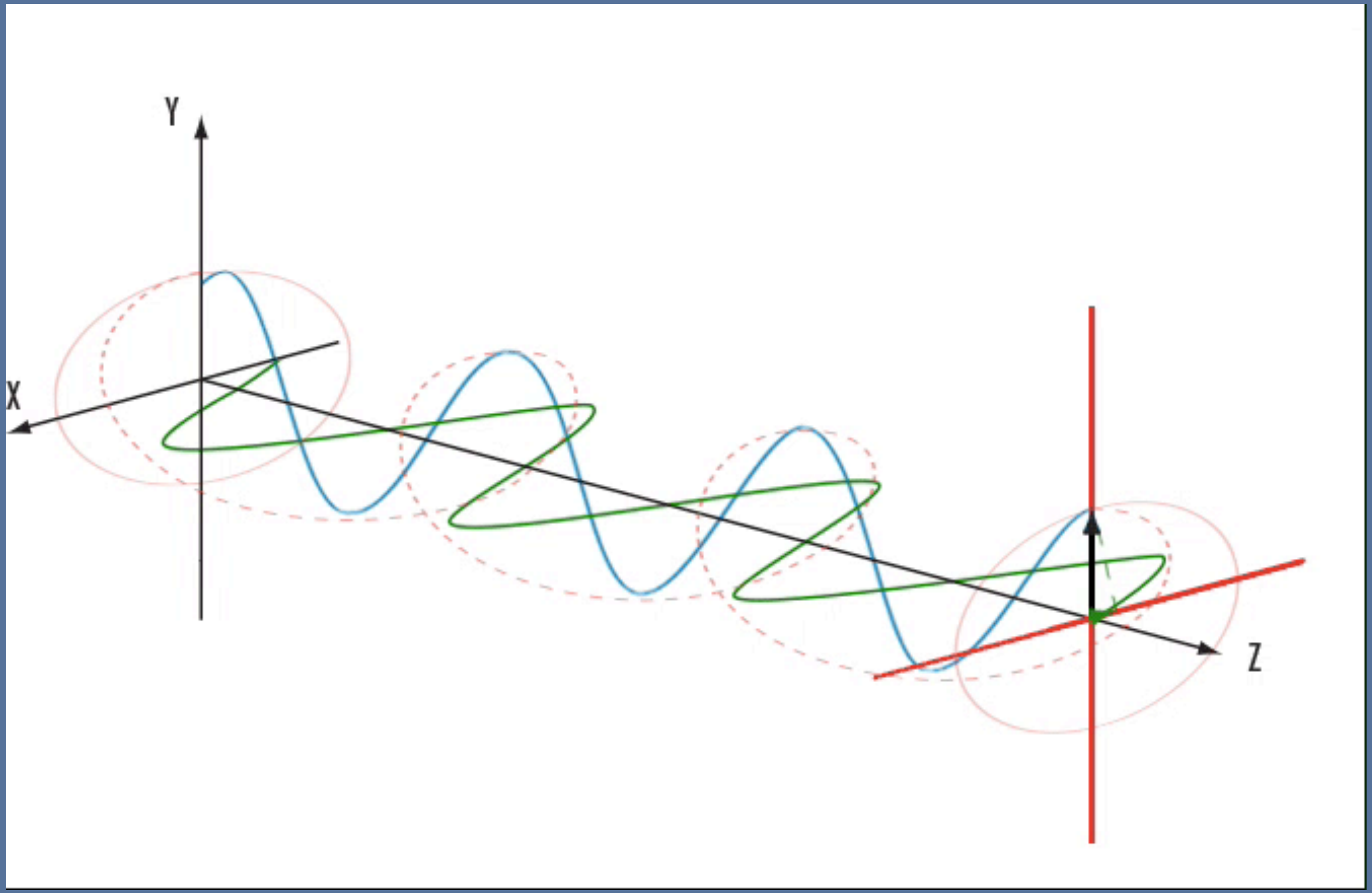
5. Stokes parameters and Stokes vector

Circularly polarised light



5. Stokes parameters and Stokes vector

Elliptically polarised light



5. Stokes parameters and Stokes vector

- For a partially polarised wave, there is a polarised contribution and a non-polarised one

$$I = I_\nu^{\rm unpol} + < E_{x0}^2 + E_{y0}^2 >$$
$$Q = < E_{x0}^2 - E_{y0}^2 >$$
$$U = < 2\, E_{x0}\, E_{y0}\, \cos\Delta >$$
$$V = < 2\, E_{x0}\, E_{y0}\, \sin\Delta >$$

Sum of the non polarised contribution and the polarised contribution, averaged over time

- For $I = V \quad (\Delta = \pi/2)$: right circular polarised wave (for an observer towards whom the wave is propagating, the tip of $\overrightarrow{E}$ moves clockwise along a circle). What are E_{x0} and E_{y0} ?
- For $I = Q$: linear polarisation along the x axis.
- For $I = U$: linear polarisation along a direction ?
- For elliptical polarisation, $E_{x0} \neq E_{y0}$ (amplitude and phase difference take any values, but remain constant)

5. Stokes parameters and Stokes vector

- To define the Stokes vector, a coordinate system has to be specified
  - The reference polarisation vector is chosen as $\overrightarrow{k}$, along the propagation direction
  - A supplementary unit vector $\vec{s}$ along y is defined. Obviously $\vec{s} \cdot \overrightarrow{k} = 0$
  - $\vec{s}$ gives the direction of $\overrightarrow{E}$ for a polarised wave with $Q = -I$ and $U = V = 0$
- To change coordinate system (e.g. rotation by an angle ψ to switch from (x, y) to (x', y'), we use the rotation matrix $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
- The new Stokes vector is given by the Müller matrix $\begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\psi & \sin 2\psi & 0 \\ 0 & -\sin 2\psi & \cos 2\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}$