Radiative Transfer

3. The formal radiative transfer equation

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Introduction

- Litterally speaking the expression “radiative transfer” is a little misleading
  - it gives the impression that we are interested in photon movement, but as we have seen at the beginning, photons propagate in a straight line, and in the absence of interaction with matter, the intensity is constant in the propagation direction of the radiation
  - In this case is radiative transfer completely trivial
- In fact, the main difficulty to solve is the interaction between radiation and matter
- The interaction between matter and radiation can add or remove radiation along the propagation direction
- In this chapter, we will look at the formalism for adding and removing radiation, derive the transfer equation and see how to solve it in simple cases

1. Extinction coefficient

1.1 Mean free path

- We consider a medium which can absorb radiation
  - A photon has a given probability of being absorbed
  - Some will be absorbed very quickly, others will be able to cross a large distance in the medium
- The efficiency of a medium to absorb photons is described by the mean free path l_{free}
- It is the mean distance travelled by a photon before being absorbed
- Unit: cm (cgs)
- l_{free} is a function
  - Of frequency: a medium which absorbs at a wavelength does not always absorb at another.
  - Of position $\overrightarrow{x}$

1.1 Mean free path

- If we consider a medium which only absorbs (i.e. no emission), with a length equal to the mean free path
  - If N_0 is the number of incident photons, the number of photons that cross the medium is N_0/e, i.e. 36.8% of photons cross the medium
  - Similarly, the intensity is also attenuated by a factor e
  - If the medium has a length of 2 mean free paths, the intensity is attenuated by a factor $e^2 = 7.4$ (only 13.5% of the photons cross the medium)
- The attenuation of the radiation is exponential with the number of mean free paths
- For denser media, the mean free paths will be shorter

1.2 Extinction coefficient

- In radiative transfer, the mean free path is rarely used
- The extinction coefficient α_ν is used instead: $\alpha_\nu = \frac{1}{l_\mathrm{free}}$
- Unit: cm^{-1} (cgs)
- This coefficient is sometimes called opacity, but this term can be confusing (it should not be confused with optical depth)
- α_ν is the extinction per unit length

1.2 Extinction coefficient

- We define also
  - The extinction cross section σ_ν (extinction coefficient per particle), in cm^2

    If n is the number density of particles [in cm^3], then $\alpha_\nu = \sigma_\nu n$

  - The mass extinction coefficient κ_ν, in $\mathrm{cm}^2\,\mathrm{g}^{-1}$

    If ρ is the mass density of particles [in $\mathrm{g\,cm}^3$], then $\kappa_\nu = \dfrac{\alpha_\nu}{\rho}$

    κ_ν is sometimes called "opacity"

1.2 Extinction coefficient

- When geometric optics apply ($\lambda \ll a$ with a the particle radius), the extinction cross section is equal to the geometric cross section, i.e. $\sigma_\nu = \pi a^2$
- In the case $a \ll \lambda$, $\quad \sigma_\nu \ll \pi a^2$
- Relation between κ_ν and σ_ν : $\kappa_\nu = \dfrac{\sigma_\nu}{m}$, with m the mass of a particle
- α_ν can also be seem as a cross section per unit volume, in $\mathrm{cm}^2\,\mathrm{cm}^{-3} = \mathrm{cm}^{-1}$
- Distinction extinction / absorption:
    - extinction is all that removes photons from the beam, and therefore includes scattering and absorption
    - We will use "extinction" only for this, and "absorption" will be used only in the case of photon destruction (some say "true absorption") for photon destruction
- Notation: some authors use κ_ν for the monochromatic absorption coefficient (and not extinction) per unit length. Watch out for the definition of the quantities!

1.2 Extinction coefficient

- What does the index ν mean in α_ν ?
  - What is the conversion factor between α_ν and α_λ?
  - And between κ_ν and κ_λ ?
  - Is it useful to define a total extinction coefficient $\alpha = \int \alpha_\nu \, d\nu$?
- For a medium that contains several types of particles, that each have their own extinction coefficient:
  - how can we define partial extinction coefficients?
  - How can we combine them to obtain a total extinction coefficient (for $\alpha_\nu, \kappa_\nu,$ and σ_ν)?

1.3 Optical depth

- Optical depth/thickness is the number of mean free paths in a medium in the direction of propagation

- It is noted τ_ν. It has no unit and no dimension

- Do not mix the optical depth and the opacity, as certain people often mistakenly use “opacity” instead of “optical depth”.

- If $\tau_\nu \ll 1$, the medium is said to be optically thin

    - In this case, the photons have no or very few interactions with the medium

- If $\tau_\nu \gg 1$, the medium is said to be optically thick

    - What we see in this case is the region where photons had their last interaction, ie the layer from which they escape. It is then possible to define a “surface”

1.3 Optical depth

- Optical depth depends on the wavelength. A medium can be optically thick at one wavelength and optically thin at another.
- Is the sun optically thick at all wavelengths?
- Link between the optical depth, the extinction coefficient, and the mean free path
  - With §1.1, we have $I = I_0 \, \mathrm{e}^{-\frac{\Delta s}{l_{\mathrm{free}}}}$, where l_{free} is the mean free path and Δs the length of the medium.
  - $\dfrac{\Delta s}{l_{\mathrm{free}}}$ is the number of mean free paths in the medium, which is the definition of τ_ν, i.e.
    $$\tau_\nu = \frac{\Delta s}{l_{\mathrm{free}}} = \alpha_\nu \, \Delta s$$
  - Over an infinitesimal length ds, the intensity changes by $dI_\nu = -I_\nu \dfrac{ds}{l_{\mathrm{free}}} = -I_\nu \, \alpha_\nu \, ds$.
    $$\Rightarrow \frac{dI_\nu}{ds} = -\alpha_\nu \, I_\nu$$

1.3 Optical depth

Optical depth vs. optical thickness

- The optical thickness is defined in the direction of propagation: $d\tau_\nu = \alpha_\nu \, ds$

- τ_ν and s increase in the same direction.

- The optical thickness between two points along the ray can be written $\tau_\nu(s_0, s_1) = \int_{s_0}^{s_1} \alpha_\nu(s) ds$, where $\alpha_\nu(s)$ is the extinction coefficient at s along the ray, between two points $\overrightarrow{x}$ and $\overrightarrow{x_0}$ such as $\overrightarrow{x} = \overrightarrow{x_0} + s\,\overrightarrow{n}$, with $\overrightarrow{n}$ along the direction of propagation

- This definition is relevant in the object's viewpoint

1.3 Optical depth

Optical depth vs. optical thickness

- The optical depth is defined in the direction opposite the direction of propagation: $d\tau_\nu = -\alpha_\nu\, ds$

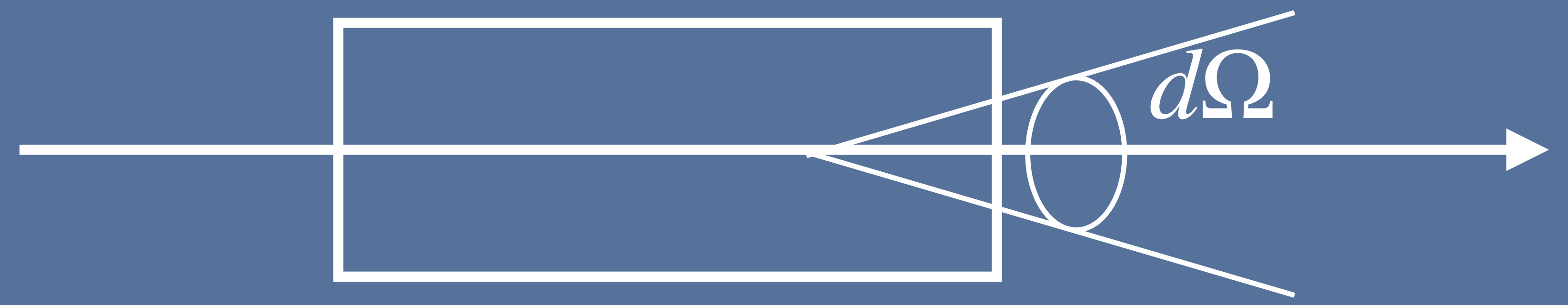
- The optical depth τ_ν increases in the opposite direction to that of s
- This definition is relevant from the observer's viewpoint
- Not all authors make the distinction

1.3 Optical depth

- Except for very particular cases (masers), $\tau_\nu > 0$
- What is the dimension of τ_ν and $d\tau_\nu$? Do optical depths add up ?
- We can also define $d\tau_\nu$ using σ_ν and κ_ν instead of α_ν
- What is the meaning of the index ν in τ_ν?
  - How can we convert τ_ν in τ_λ?
  - What is the meaning of $\displaystyle\int_0^\infty \tau_\nu \, d\nu$?
- What is the optical depth of a homogeneous model of length D and of mean free path l_ν?

2. Emissivity

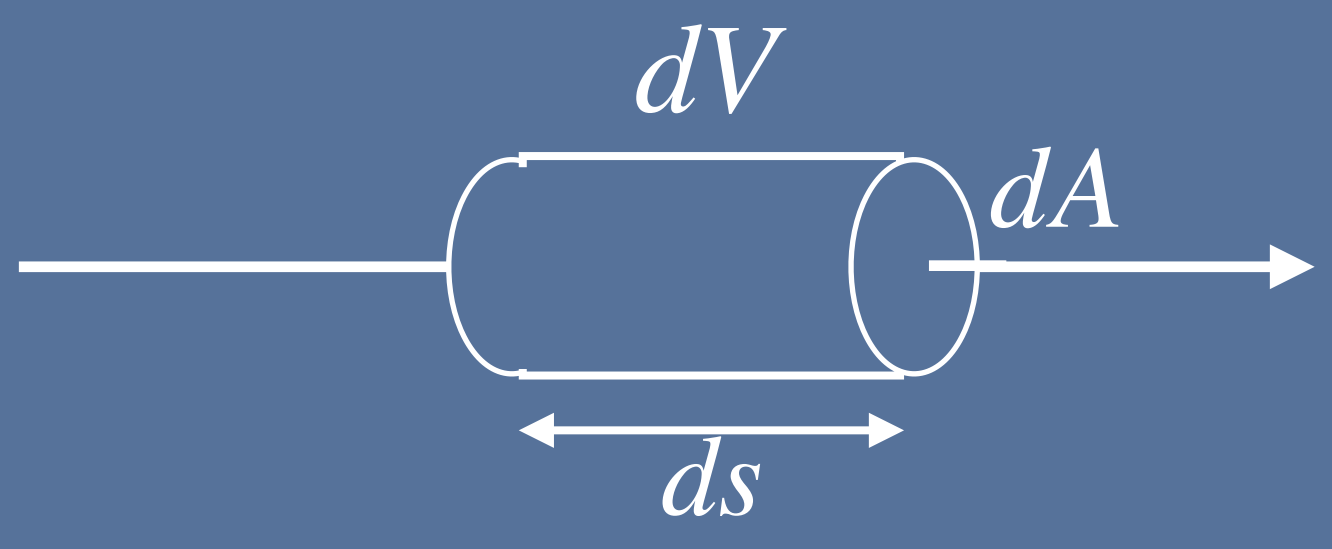
- We consider a medium that can emit electromagnetic radiation, ie that can add photons along the direction of propagation.
- The number of added photons, ie the added energy, is proportional to the number of emitting particles, to the time interval dt, to the bandwidth interval $d\nu$, and to the emission solid angle $d\Omega$



- The proportionality coefficient is called the emissivity and is noted j_ν . It is defined either per unit volume (as in this lecture) or per unit mass.
- $dE_\nu = j_\nu \, dV \, dt \, d\nu \, d\Omega$
- Unit: $erg \, s^{-1} \, cm^{-3} \, Hz^{-1} \, sr^{-1}$
- The emissivity depends on the location, time and frequency, like the intensity I_ν

2. Emissivity

- The emission volume dV is equal to the product of the section dA and the path ds: $dV = dA\, ds$



- Combining the expression of the intensity seen in Chapter 2, $dE_\nu = I_\nu\, dA\, dt\, d\nu\, d\Omega$, and the definition of j_ν, $dE_\nu = j_\nu\, dV\, dt\, d\nu\, d\Omega$, we obtain:
- $dI_\nu = j_\nu(s)\ ds$ for an only-emitting medium
- It is the intensity added along the optical path by the local photon emission
- Note that j_ν is sometimes written ϵ_ν

2. Emissivity

- Why is the emission coefficient defined in terms of intensity and not in terms of flux?
- For two types of particules or emission processes, what is the total emission coefficient (at the same frequency)?

3. Formal transfer equation

3.1 General form

- In the previous form, we had written that the radiative energy remained constant if there was no interaction with the medium: $\dfrac{dI_\nu(\overrightarrow{n}, s)}{ds} = 0$
- If there are interactions with the medium this equation is modified:
  - By an extinction term: $\dfrac{dI_\nu(\overrightarrow{n}, s)}{ds} = -\alpha_\nu(s)\, I_\nu(\overrightarrow{n}, s)$ (s is the coordinate along the ray)

    This is the formal radiative transfer equation for a pure extincting medium (not emitting). The equation is valid along a ray, for any ray that crosses the medium
  - By an emission term: $\dfrac{dI_\nu(\overrightarrow{n}, s)}{ds} = j_\nu(s)$

3.1 General form

- Adding these two terms we obtain $\dfrac{dI_\nu(\overrightarrow{n}, s)}{ds} = j_\nu(s) - \alpha_\nu(s)\, I_\nu(\overrightarrow{n}, s)$ which is the formal radiative transfer equation

  s is the coordinate along $\overrightarrow{n}$, the direction of propagation

- Vector form of the radiative transfer equation:
  $\overrightarrow{n} \cdot \overrightarrow{\nabla}\, I_\nu(\overrightarrow{x}, \overrightarrow{n}) = j_\nu(\overrightarrow{x}, \overrightarrow{n}) - \alpha_\nu(\overrightarrow{x})\, I_\nu(\overrightarrow{x}, \overrightarrow{n})$

3.1 General form

- The radiative transfer equation can be written with the optical depth

$$\frac{dI_\nu(\vec{n}, s)}{ds} = j_\nu(s) - \alpha_\nu(s)\, I_\nu(\vec{n}, s)$$

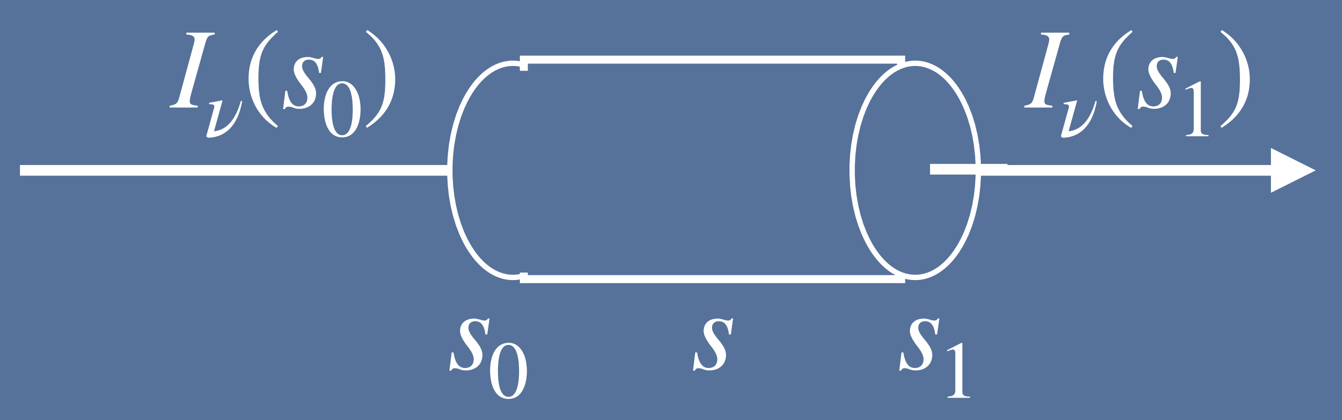
- with $d\tau_\nu = \alpha_\nu\, ds$, this yields: $$\frac{dI_\nu(\vec{n}, s)}{d\tau_\nu} = \frac{j_\nu(s)}{\alpha_\nu(s)} - I_\nu(\vec{n}, s)$$
- And with $d\tau_\nu = -\alpha_\nu\, ds$: $$\frac{dI_\nu(\vec{n}, s)}{d\tau_\nu} = I_\nu(\vec{n}, s) - \frac{j_\nu(s)}{\alpha_\nu(s)}$$

3.2 Integral expression

- In the absence of emission, the equation is $\dfrac{dI_\nu(\overrightarrow{n}, s)}{ds} = -\,\alpha_\nu(s)\, I_\nu(\overrightarrow{n}, s)$, which by integration gives

  $$I_\nu(\overrightarrow{n}, s_1) = I_\nu(\overrightarrow{n}, s_0)\, \mathrm{e}^{-\tau_\nu(s_0, s_1)}$$

- If the medium extends from s_0 to s_1 along the direction of propagation, with $\tau_\nu(s_0, s_1)$ the optical depth from s_0 to s_1.



The medium is not necessarily cylindrical, this is just an illustration

3.2 Integral expression

- For emission in a medium that does not absorb, the (formal) solution is trivial:

$$I_\nu(\vec{n}, s_1) = \int_{s_0}^{s_1} j_\nu(s)\, ds$$

- For emission in an extincting medium, we have to take into account the attenuation of the signal between s and s_1:

$$I_\nu(\vec{n}, s_1) = \int_{s_0}^{s_1} j_\nu(s)\; \mathrm{e}^{-\tau_\nu(s, s_1)}\, ds$$

- So that adding both contributions, the formal transfer equation is

$$I_\nu(\vec{n}, s_1) = I_\nu(\vec{n}, s_0)\, \mathrm{e}^{-\tau_\nu(s_0, s_1)} + \int_{s_0}^{s_1} j_\nu(s)\; \mathrm{e}^{-\tau_\nu(s, s_1)}\, ds$$, where little has been solved, really

4. Source function

- In the expression of the radiative transfer equation with the optical depth, the term j_ν/α_ν has appeared.

- For reason that will become clear in the next section, we define the source function S_ν, which is

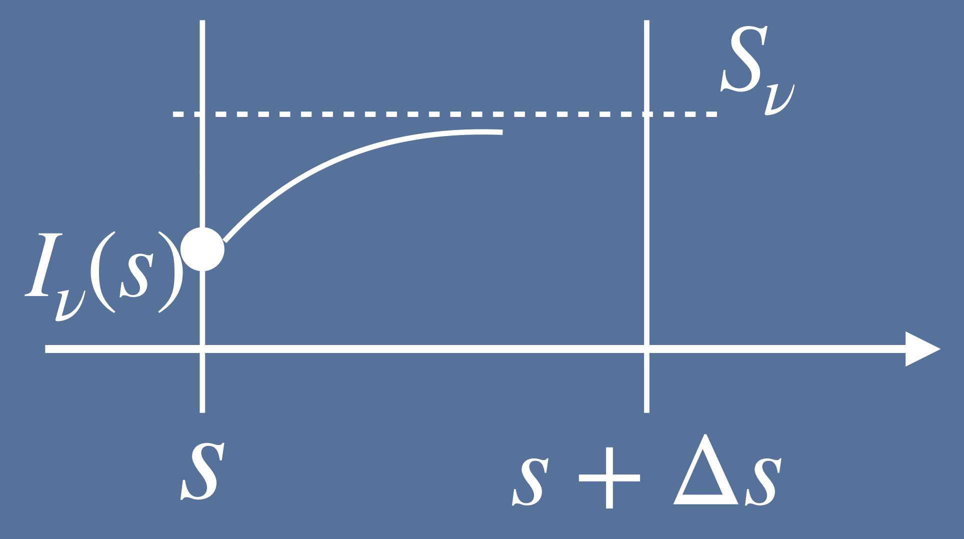
$$S_\nu = \frac{j_\nu}{\alpha_\nu}$$

- The transfer equation becomes then

$$\frac{dI_\nu(\vec{n}, s)}{ds} = \alpha_\nu(s)\,[S_\nu(s) - I_\nu(\vec{n}, s)]$$

- Interpretation: the source function acts as an “attractor” for the intensity: at each point along the ray, the intensity tends towards S_ν in the course of the propagation. If S_ν is constant along the ray, the intensity reaches S_ν asymptotically after a few mean free paths.

4. Source function



- In the constant case:

$$\frac{I_\nu(s+\Delta s) - I_\nu(s)}{\Delta s} = \alpha_\nu(s)\,[S_\nu - I_\nu(s)] \Rightarrow I_\nu(s+\Delta s) = \alpha_\nu(s)\Delta s\,[S_\nu - I_\nu(s)] + I_\nu(s)$$

At each step, $I_\nu(s)$ comes closer to S_ν

- If S_ν is not constant, I_ν is always “late” but will try to approach S_ν along the direction of propagation

4. Source function

- This integral expression can also be derived from the transfer equation with the optical depth. We will assume $s_0 = 0$ for convenience.

$\dfrac{dI_\nu}{d\tau_\nu} = S_\nu(s) - I_\nu(s)$, in which we multiply each term by e^τ and integrate:

$$\int_0^{\tau_\nu(s)} e^\tau \frac{dI_\nu}{d\tau}\, d\tau = \int_0^{\tau_\nu(s)} e^\tau \left[S_\nu(s) - I_\nu(s)\right] d\tau$$

$$\Big[e^\tau\, I_\nu\Big]_0^{\tau_\nu(s)} - \int_0^{\tau_\nu(s)} e^\tau\, I_\nu\, d\tau = \int_0^{\tau_\nu(s)} e^\tau\, S_\nu\, d\tau - \int_0^{\tau_\nu(s)} e^\tau\, I_\nu\, d\tau$$

$$e^{\tau_\nu(s)}\, I_\nu(s) - I_\nu(0) = \int_0^{\tau_\nu(s)} e^\tau\, S_\nu\, d\tau$$

$$\Rightarrow \qquad I_\nu(s) = e^{-\tau_\nu(s)}\, I_\nu(0) + \int_0^{\tau_\nu(s)} e^{-(\tau_\nu(s)-\tau)}\, S_\nu\, d\tau$$

4. Source function

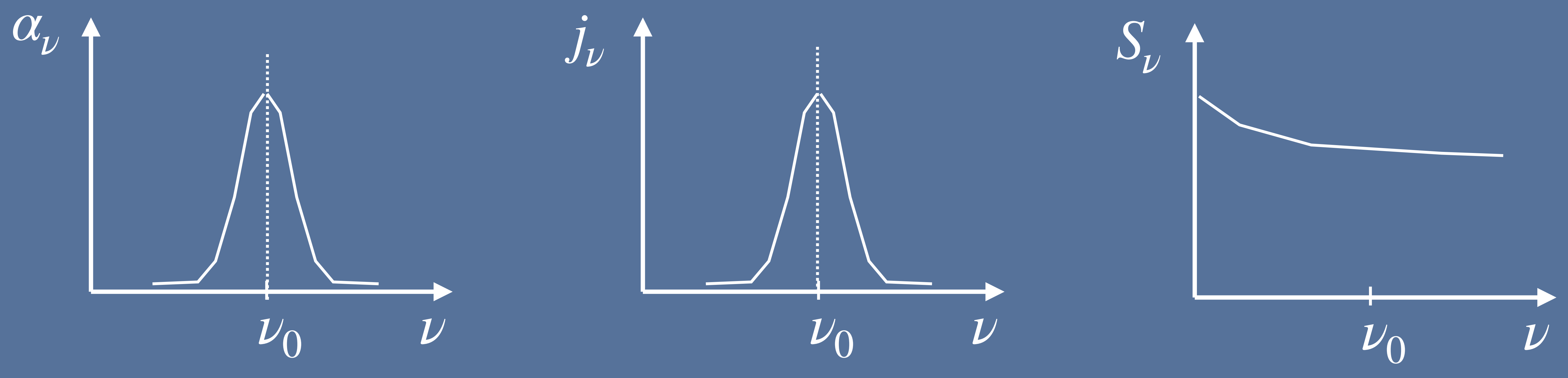
- We can also define the source function with σ_ν and κ_ν for the extinction coefficient
  - $S_\nu = \dfrac{j_\nu}{\alpha_\nu} = \dfrac{j_\nu}{\sigma_\nu\, n} = \dfrac{j_\nu}{\kappa_\nu\, \rho}$
- If different processes contribute to emission and extinction at frequency ν, how can we define the total source function in terms of the individual source functions attached to each process?
  - $j_\nu^{\rm tot} = j_\nu^{\rm A} + j_\nu^{\rm B}$ $\qquad\qquad$ $\alpha_\nu^{\rm tot} = \alpha_\nu^A + \alpha_\nu^B$
  - $S_\nu^{\rm tot} = \dfrac{j_\nu^{\rm tot}}{\alpha_\nu^{\rm tot}} = \dfrac{\alpha_\nu^A\, S_\nu^A + \alpha_\nu^B\, S_\nu^B}{\alpha_\nu^A + \alpha_\nu^B}$, where $S_\nu^A = \dfrac{j_\nu^A}{\alpha_\nu^A}$ and $S_\nu^B = \dfrac{j_\nu^B}{\alpha_\nu^B}$

4. Source function

- Three quantities are used, j_ν, α_ν and S_ν to describe the addition and subtraction of intensity along the direction of propagation.

- Most often, α_ν and S_ν are used instead of j_ν and α_ν. There are two reasons for this
  - We can then have a "symmetric" transfer equation $\dfrac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$
  - α_ν and S_ν tend to be much more independent of one another than j_ν and α_ν.

For a bound-bound transition



Emissivity and absorption are linked. Both peak at the line frequency but S_ν is a much smoother function. Both peaks nearly cancel out

4. Source function

Application

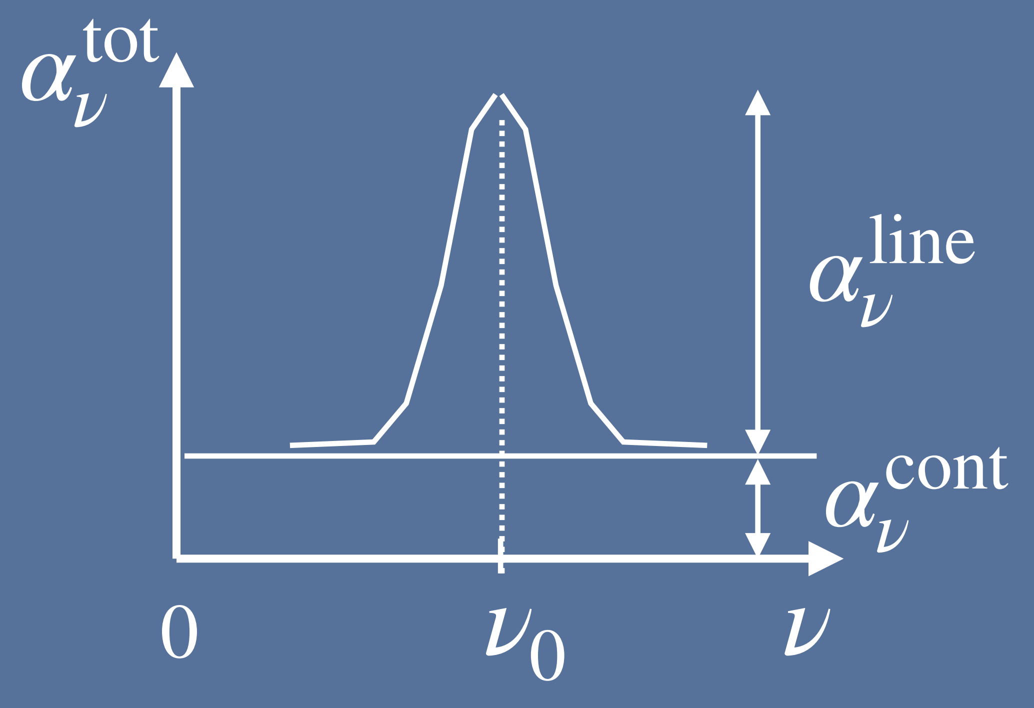
- For a bound-bound transition, $j_\nu^{\rm line}$ and $\alpha_\nu^{\rm line}$ both vary quickly
  - What is the total source function if there is additionally an emission $j_\nu^{\rm cont}$ and an absorption $\alpha_\nu^{\rm cont}$ at the frequency of the line?
  - When do we have $S_\nu^{\rm total} \simeq S_\nu^{\rm line}$ and $S_\nu^{\rm total} \simeq S_\nu^{\rm cont}$?
  - Show that $S_\nu^{\rm total}$ hardly varies over the linewidth if $S_\nu^{\rm line} \simeq S_\nu^{\rm cont}$

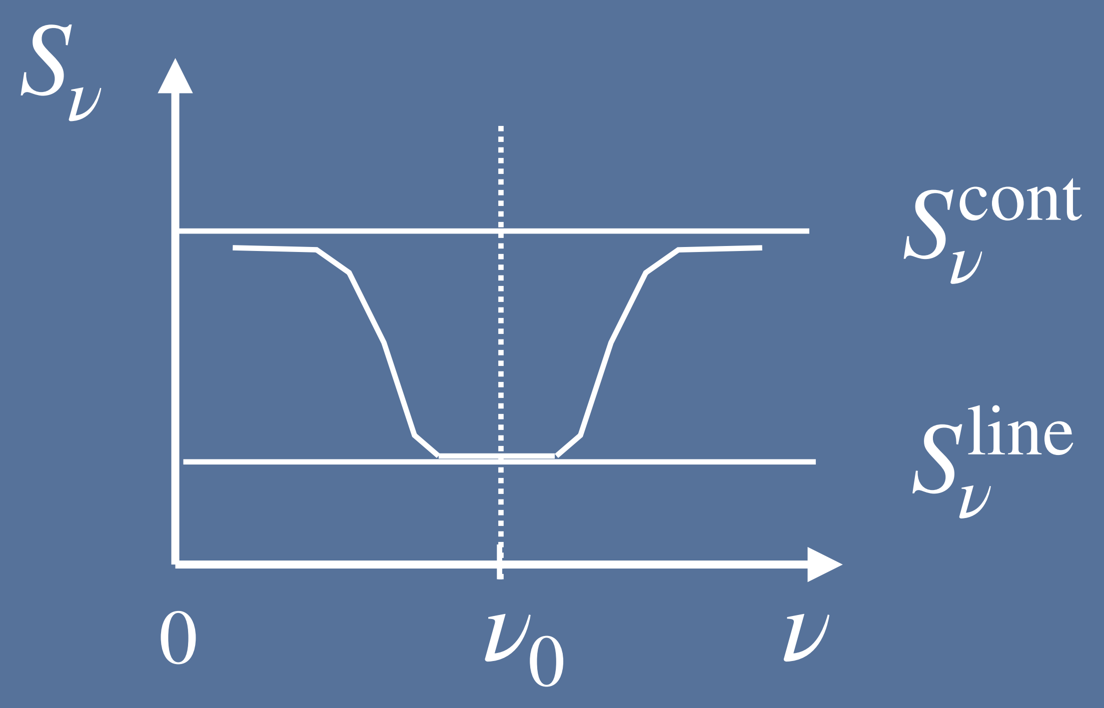
4. Source function

- Individual source functions: $S_\nu^{\rm line} = \dfrac{j_\nu^{\rm line}}{\alpha_\nu^{\rm line}}$ and $S_\nu^{\rm cont} = \dfrac{j_\nu^{\rm cont}}{\alpha_\nu^{\rm cont}}$

- Total source function: $S_\nu^{\rm tot} = \dfrac{\alpha_\nu^{\rm line}\, S_\nu^{\rm line} + \alpha_\nu^{\rm cont}\, S_\nu^{\rm cont}}{\alpha_\nu^{\rm line} + \alpha_\nu^{\rm cont}} = \dfrac{S_\nu^{\rm cont} + \eta_\nu\, S_\nu^{\rm line}}{1 + \eta_\nu}$

  if we denote $\eta_\nu = \dfrac{\alpha_\nu^{\rm line}}{\alpha_\nu^{\rm cont}}$





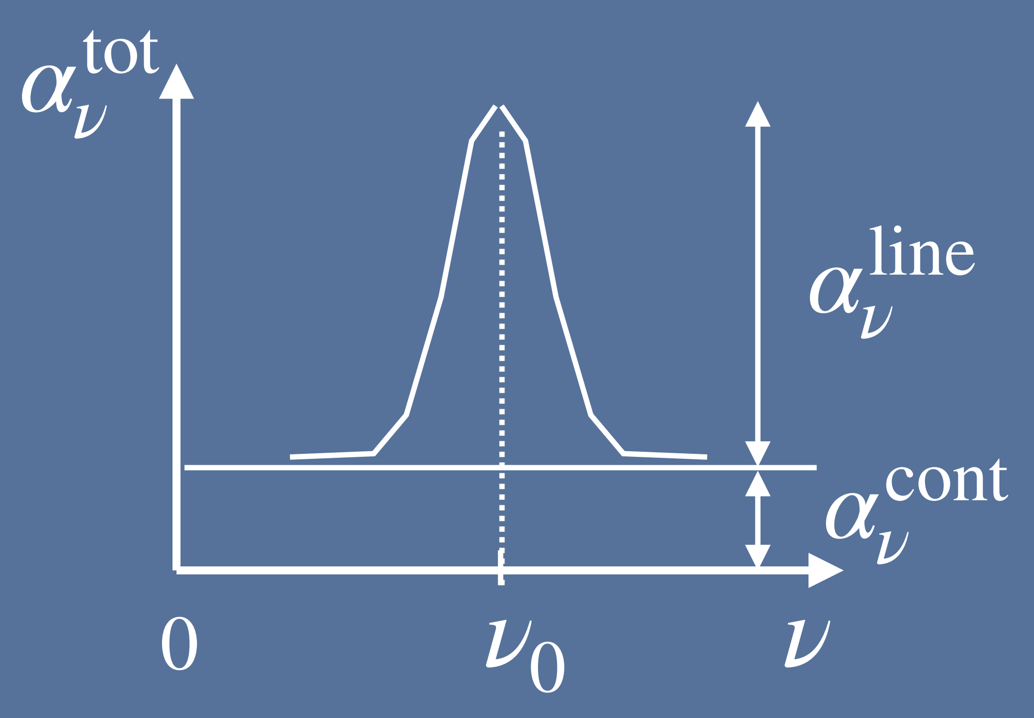
If $\eta_\nu \gg 1$, $S_\nu^{\rm tot} \sim S_\nu^{\rm line}$

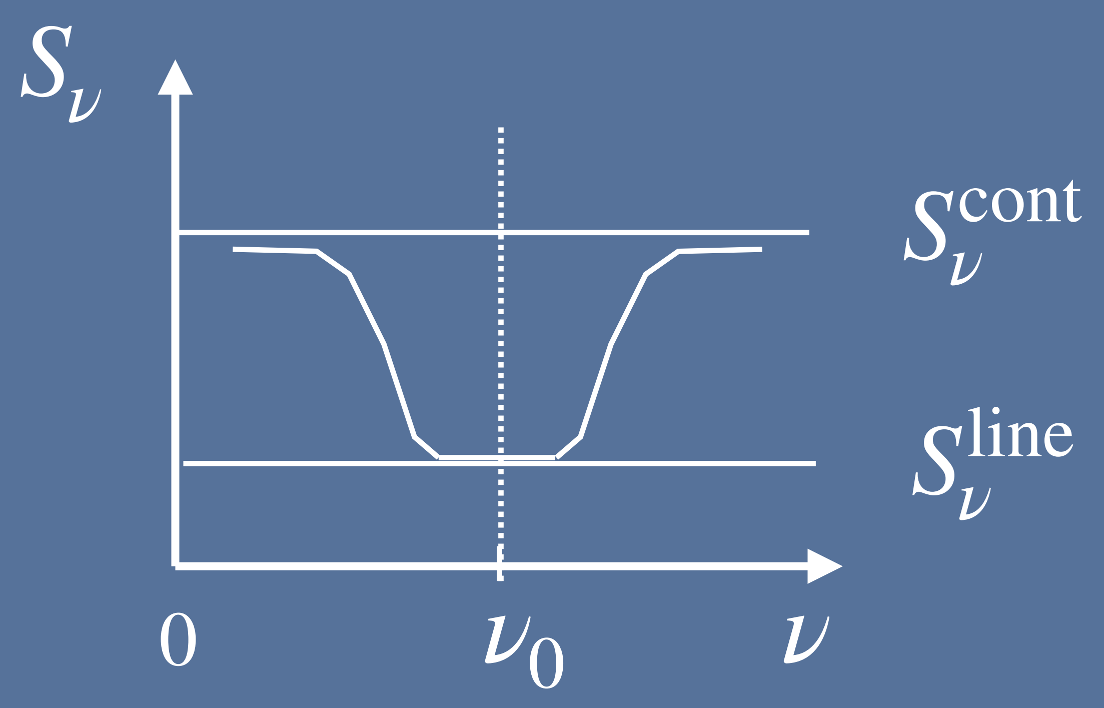
If $\eta_\nu \ll 1$, $S_\nu^{\rm tot} \sim S_\nu^{\rm cont}$

If $S_\nu^{\rm line} \neq S_\nu^{\rm cont}$, $S_\nu^{\rm tot}$ depends on frequency

4. Source function

- Far from the line, $\alpha_\nu^{\text{line}} \ll \alpha_\nu^{\text{cont}}$ so that $S_\nu^{\text{tot}} \simeq S_\nu^{\text{line}}$
- On the line, $\alpha_\nu^{\text{line}} \gg \alpha_\nu^{\text{cont}}$ and $S_\nu^{\text{tot}} \simeq S_\nu^{\text{cont}}$
- If $S_\nu^{\text{line}} \sim S_\nu^{\text{cont}}$, the variations of S_ν^{tot} are very small: both straight lines at $S_\nu = S_\nu^{\text{line}}$ and $S_\nu = S_\nu^{\text{cont}}$ overlap, and S_ν^{tot} “oscillates” between both lines, ie S_ν^{tot} does not depend on frequency
- If $S_\nu^{\text{line}} \neq S_\nu^{\text{cont}}$, S_ν^{tot} varies with frequency even if S_ν^{line} doesn’t, because η_ν follows the variations of α_ν





If $\eta_\nu \gg 1$, $S_\nu^{\text{tot}} \sim S_\nu^{\text{line}}$

If $\eta_\nu \ll 1$, $S_\nu^{\text{tot}} \sim S_\nu^{\text{cont}}$

If $S_\nu^{\text{line}} \neq S_\nu^{\text{cont}}$, S_ν^{tot} depends on frequency

4. Source function

- Assuming that there is no photon creation, destruction or conversion (ie j_ν and α_ν only depend on monochromatic scattering), what is the source function?
  - With scattering (assumed to be isotropic and elastic), photons only change direction
  - Photons scattered out of the beam (losses): $dI_\nu = \alpha_\nu I_\nu ds$
  - Photons scattered into the beam: $dI_\nu = j_\nu ds$
  - If we assume time invariability, at each location the total emission in all directions has to be equal to the total extinction in all directions: $\int j_\nu \, d\Omega = \int \alpha_\nu I_\nu \, d\Omega$
  - By definition, $J_\nu = 1/4\pi \int I_\nu \, d\Omega$, and assuming isotropy, we obtain

$$j_\nu = \alpha_\nu J_\nu \quad \Rightarrow \quad S_\nu^{\mathrm{sca}} = \frac{j_\nu}{\alpha_\nu} \quad \Rightarrow \quad S_\nu^{\mathrm{sca}} = J_\nu$$

4. Source function

- Extinction of the radiation at visible wavelengths in the Earth atmosphere mostly depends on Rayleigh scattering.
  - What is the corresponding source function ?
  - This is a similar situation as in the previous example: $\int j_\nu \, d\Omega = \int \alpha_\nu I_\nu \, d\Omega$
  - The integration of the right term is essentially over the solid angle subtended by the Sun (the contribution of other directions comes from solar photons which have already been scattered and is therefore much weaker)
  - The result is the same as before
- What is the meaning of $S_\nu = 1$? And $S_\nu / I_\nu = 1$? Is it possible to have $S_\nu > I_\nu$? and $S_\nu < 0$?

5. Solution of the transfer equation in simple cases

- We are going to derive solutions of the transfer equation in particularly simple cases
- These cases are widely used, even when it is not always justified and when they are just coarse approximations
- These (often trivial) “resolution methods” were the only ones at our disposal before the advent of powerful calculators and the development numerical methods
- These approximations concern the geometry (e.g. plane parallel), the medium (e.g. homogeneous), the coupling between matter and radiation (e.g. thermodynamic equilibrium, in the next chapter)
- One has to bear in mind that running a complex model (often time consuming) is not always better, and depends on how many constraints we have: if we want to determine a molecular abundance from a single spectrum, we will not get a better result by running a 3D radiative transfer model.

5.1 Homogeneous medium

- In a homogeneous medium, neither j_ν nor α_ν vary in space
- As a consequence, the source function S_ν is also spatially invariant
- We start from the integral form of radiative transfer:
  $I_\nu(\overrightarrow{n}, s_1) = I_\nu(\overrightarrow{n}, s_0)\, \mathrm{e}^{-\tau_\nu(s_0,s_1)} + \int_{s_0}^{s_1} j_\nu(s)\, \mathrm{e}^{-\tau_\nu(s,s_1)}\, ds$, where we use the fact that $j_\nu = \alpha_\nu\, S_\nu$ are independent of s.
- $I_\nu(\overrightarrow{n}, s_1) = I_\nu(\overrightarrow{n}, s_0)\, \mathrm{e}^{-\tau_\nu(s_0,s_1)} + \alpha_\nu\, S_\nu \int_{s_0}^{s_1} \mathrm{e}^{-\tau_\nu(s,s_1)}\, ds$
- The last term can be evaluated as follows

5.1 Homogeneous medium

- $$\int_{s_0}^{s_1} \mathrm{e}^{-\tau_\nu(s,s_1)}\, ds = \int_{s_0}^{s_1} \mathrm{e}^{-\int_s^{s_1} \alpha_\nu\, ds'}\, ds \text{ (definition of } \tau_\nu)$$

$$= \int_{s_0}^{s_1} \mathrm{e}^{-\alpha_\nu \int_s^{s_1} ds'}\, ds \quad (\alpha_\nu \text{ independent of } s)$$

$$= \int_{s_0}^{s_1} \mathrm{e}^{-\alpha_\nu (s_1 - s)}\, ds$$

$$= \left[\frac{\mathrm{e}^{-\alpha_\nu (s_1 - s)}}{\alpha_\nu} \right]_{s_0}^{s_1}$$

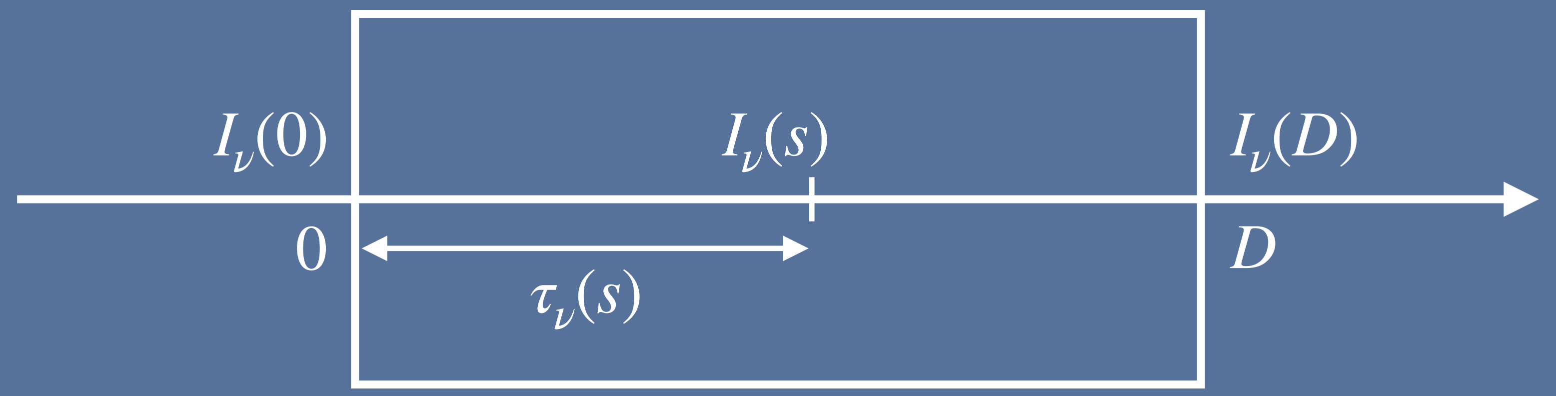
$$= \frac{1}{\alpha_\nu} \left(1 - \mathrm{e}^{-\alpha_\nu (s_1 - s_0)}\right)$$

5.1 Homogeneous medium

- Expression of $I_\nu(\overrightarrow{n}, s)$ for a homogeneous medium:
  $I_\nu(\overrightarrow{n}, s_1) = I_\nu(\overrightarrow{n}, s_0)\, \mathrm{e}^{-\tau_\nu(s_0,s_1)} + S_\nu \left(1 - \mathrm{e}^{-\tau_\nu(s_0,s_1)}\right)$

- This is a very widely used expression, also with the form
  $I_\nu(s) = I_\nu(0)\, \mathrm{e}^{-\tau_\nu(s)} + S_\nu \left(1 - \mathrm{e}^{-\tau_\nu(s)}\right)$

  where $I_\nu(0)$ is the background intensity, and τ_ν is the optical depth of the medium (total optical depth)



5.1 Homogeneous medium

- Optically thick medium: $\tau_\nu(s) \gg 1$

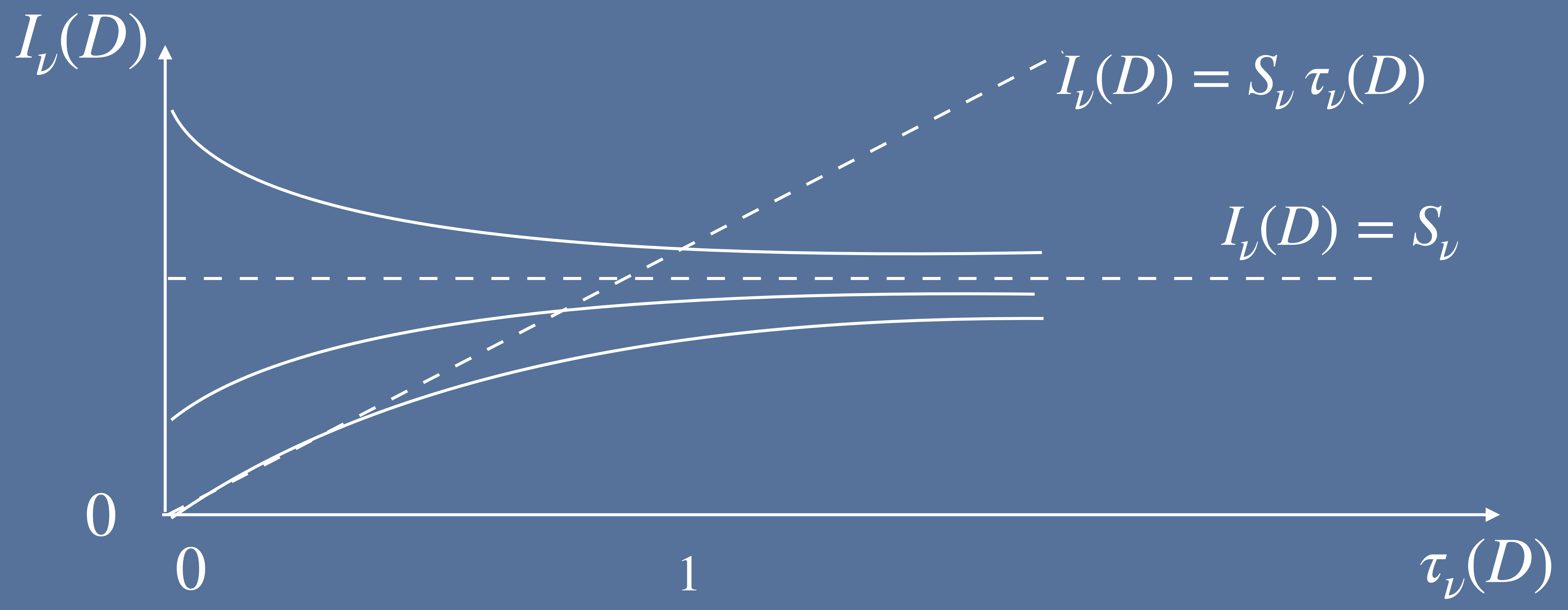
$$I_\nu(s) \sim S_\nu$$

- Optically thin medium: $\tau_\nu(s) \ll 1$

$$I_\nu(s) \sim I_\nu(0) - \tau_\nu(s)\, I_\nu(0) + \tau_\nu(s)\, S_\nu = I_\nu(0) + \tau_\nu(s)\, [S_\nu - I_\nu(0)]$$

5.1 Homogeneous medium

$I_\nu(D)$ as a function of the medium's optical depth



- For a non-illuminated optically thin object: $I_\nu(D) = S_\nu \, \tau_\nu(D)$
- If $I_\nu(0) \neq 0$, the intensity is increased with respect to the above case
- If $I_\nu(0) > S_\nu$, the intensity decreases towards the source function
- In the optically thick case, the intensity tends towards $I_\nu(D) \sim S_\nu$, independent of $I_\nu(0)$

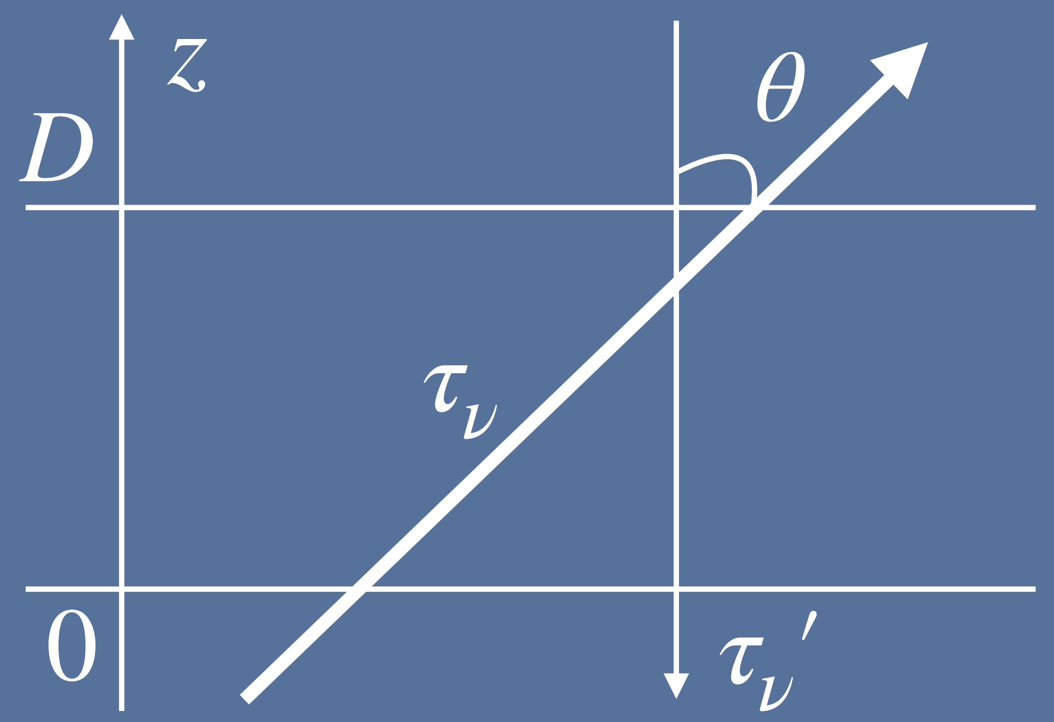
5.1 Homogeneous medium

- What is the outgoing intensity for a semi-infinite homogeneous medium
  - How does it depend on the viewing angle θ?
  - What is the intensity in an infinite homogeneous medium?
  - Why are these intensities independent of the amount of extinction in the medium?
  - Are they independent of its nature?

5.1 Homogeneous medium

- We now consider a plane parallel medium, and the intensity along a beam tilted with respect to the medium of thickness D. The inclination is $\mu = \cos\theta$.

- τ_ν is the optical thickness along the beam, and $\tau_\nu{}'$ is the optical depth perpendicular to the medium



The intensity along the beam was

$$I_\nu(s) = I_\nu(0)\, \mathrm{e}^{-\tau_\nu(s)} + S_\nu \left(1 - \mathrm{e}^{-\tau_\nu(s)}\right)$$

$$\tau_\nu(z = D) = \alpha_\nu \int_0^D \frac{dz}{\mu} = \frac{\alpha_\nu D}{\mu}$$

For the optical depth, the origin for $\tau_\nu{}'$ is for $z = D$: $\tau_\nu{}'(z = D) = 0$

A radial (⊥) beam has $\tau_\nu{}'(z = 0) = \tau_\nu(D) \quad (\mu = 1)$

The equation is: $I_\nu(D) = I_\nu(0)\, \mathrm{e}^{-\tau'_\nu(0)} + S_\nu \left(1 - \mathrm{e}^{-\tau'_\nu(0)}\right)$

For a beam inclined by μ, the path is longer by a factor $1/\mu$

$$I_\nu(D, \mu) = I_\nu(0,\mu)\, \mathrm{e}^{-\tau'_\nu(0)/\mu} + S_\nu \left(1 - \mathrm{e}^{-\tau'_\nu(0)/\mu}\right)$$

5.1 Homogeneous medium

- A homogeneous medium contains particles that produce a continuous emission $j_\nu^{\rm cont}$ and extinction $\alpha_\nu^{\rm cont}$ at frequency ν_0 and also particles that produce a bound-bound emission $j_\nu^{\rm line}$ and extinction $\alpha_\nu^{\rm line}$ centred at ν_0.
- Both corresponding source functions are equal: $S_\nu^{\rm cont} = S_\nu^{\rm line}$
- What is the outgoing intensity at the line frequency in the following cases - In each case is the line in emission or in absorption

(a) $\tau_\nu(D) \gg 1$

(b) $\tau_\nu(D) \ll 1$ and $I_\nu(0) = 0$

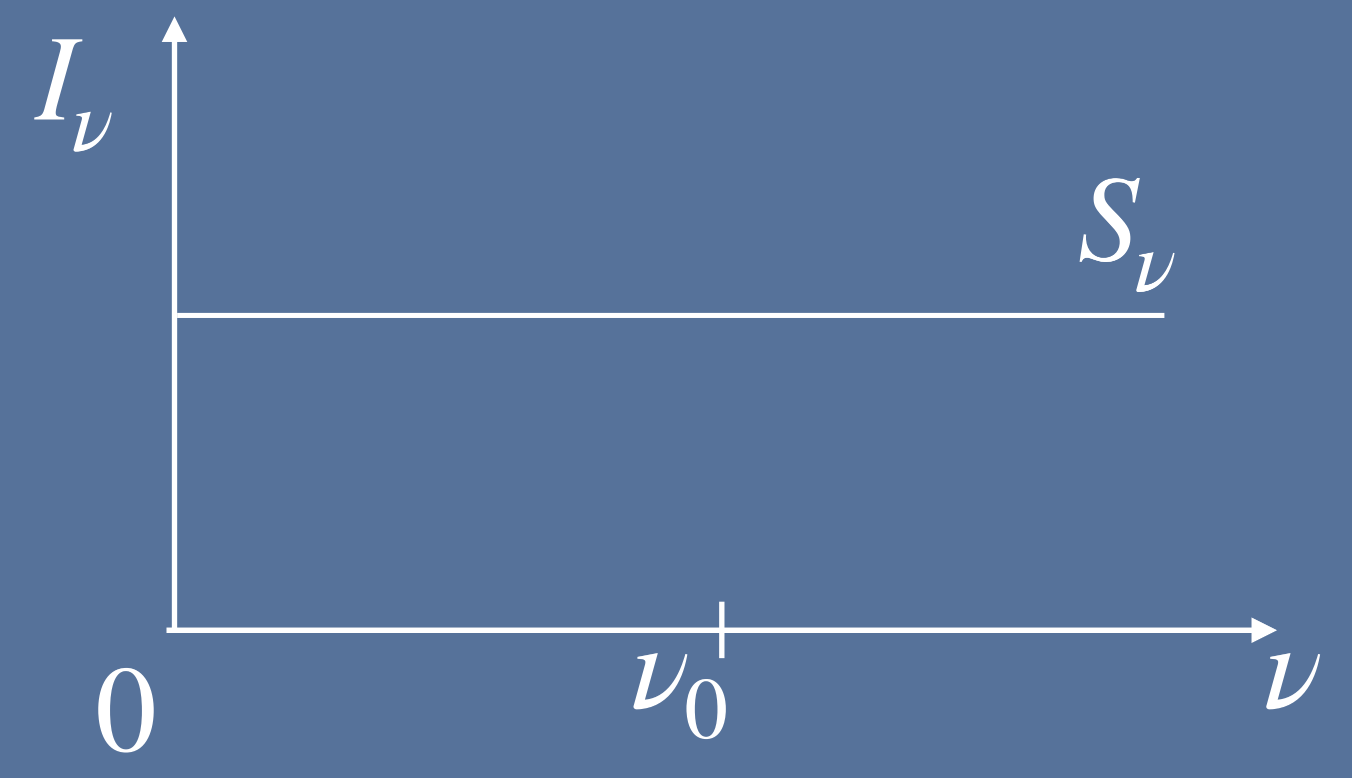
(c) $\tau_\nu(D) \ll 1$ and $I_\nu(0) < S_\nu^{\rm tot}$

(d) $\tau_\nu(D) \ll 1$ and $I_\nu(0) > S_\nu^{\rm tot}$

5.1 Homogeneous medium

- Because $S_\nu^{\rm cont} = S_\nu^{\rm line}$, $S_\nu^{\rm tot}$ varies very little and on the line we have $S_\nu^{\rm tot} \sim S_\nu^{\rm cont}$. The variations of the intensity as function of frequency only come from α_ν.

(a) $I_\nu(D) = S_\nu$ (optically thick case). In this case, there is no line because of the homogeneity of the medium ($S_\nu^{\rm tot}$ hardly varies with frequency)



5.1 Homogeneous medium

(b) $I_\nu(D) = (\alpha_\nu^{\rm cont} + \alpha_\nu^{\rm line})\, D\, S_\nu^{\rm tot}$

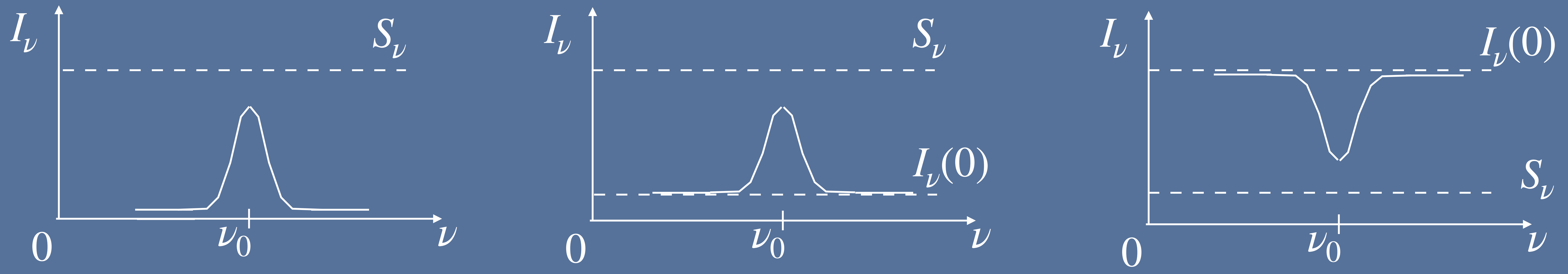
(c) $I_\nu(D) = I_\nu(0) + [S_\nu - I_\nu(0)]\,(\alpha_\nu^{\rm cont} + \alpha_\nu^{\rm line})\, D$

(d) $I_\nu(D) = I_\nu(0) - [I_\nu(0) - S_\nu]\,(\alpha_\nu^{\rm cont} + \alpha_\nu^{\rm line})\, D$

(b) Emission line

(c) Emission line

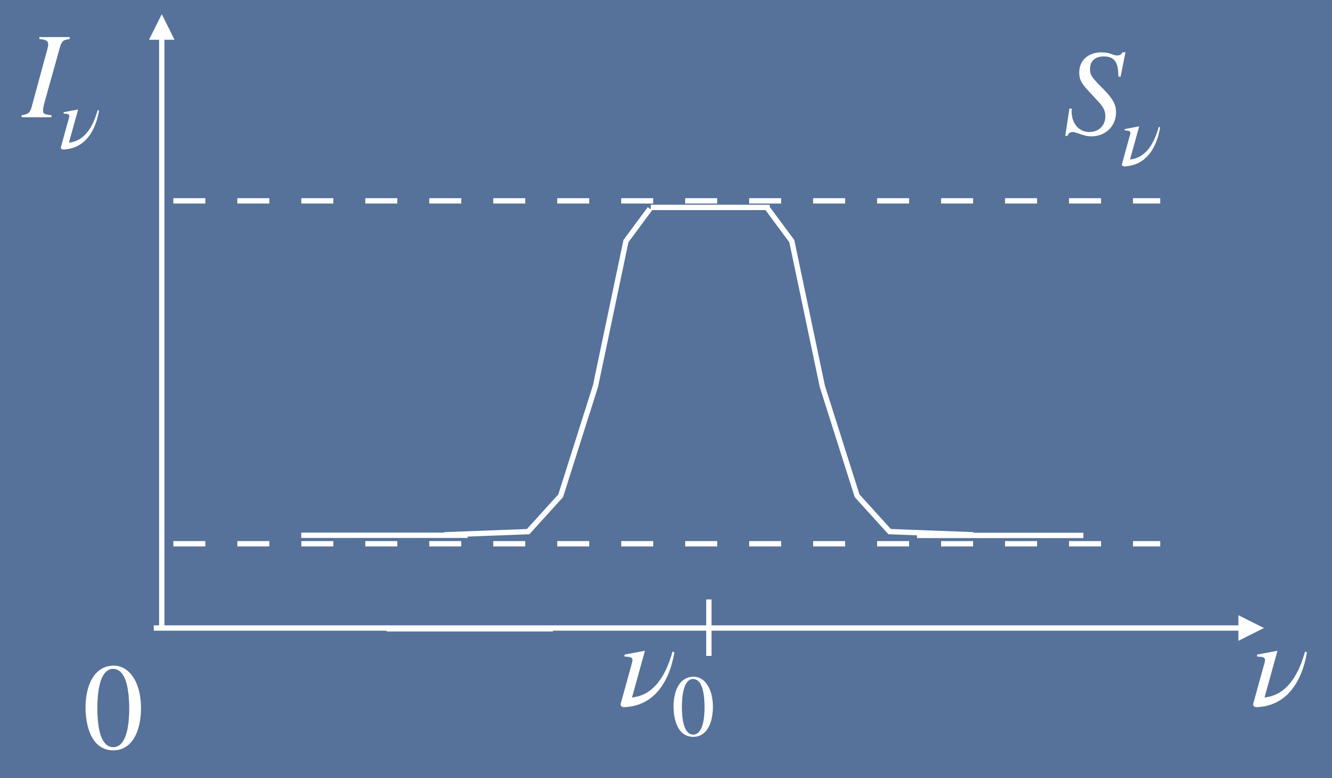
(d) Absorption line



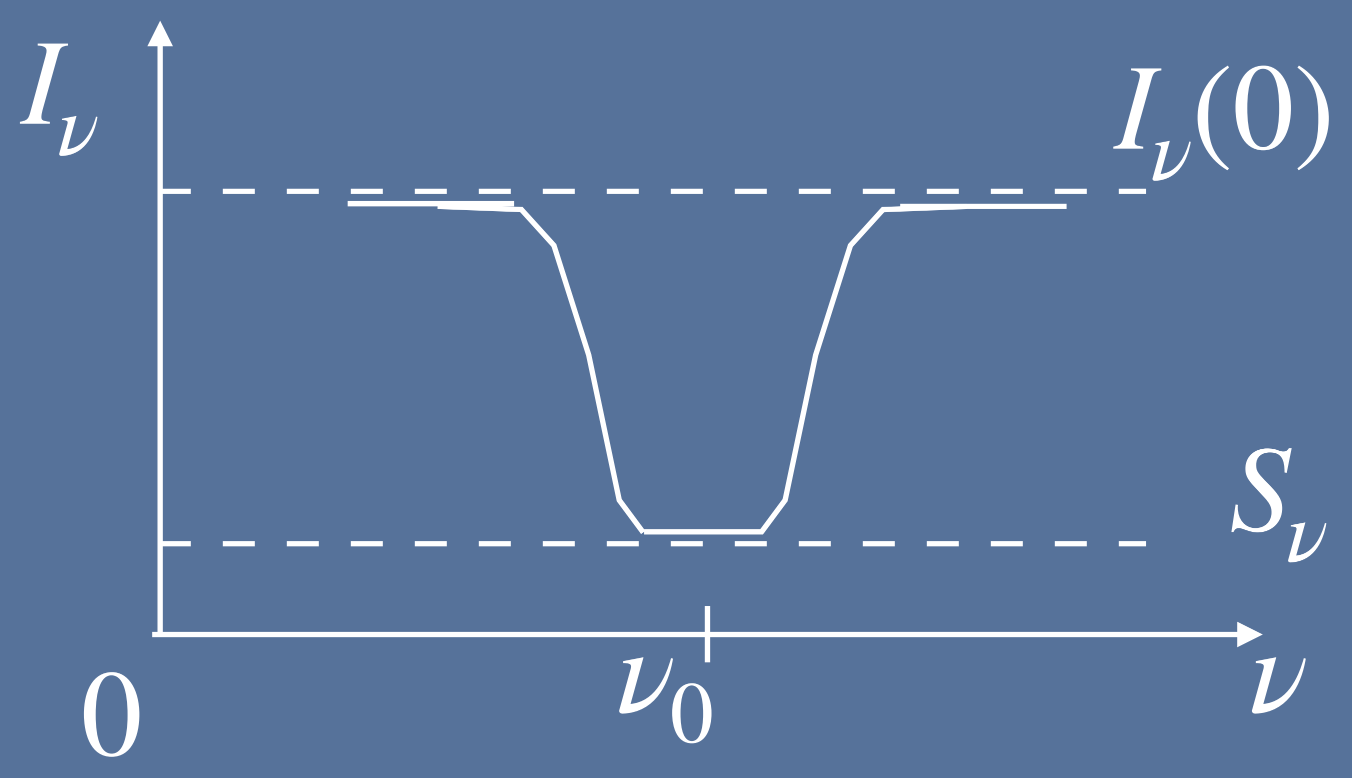
5.1 Homogeneous medium

In the case the optical depth in ν_0 is large (ie $\tau_{\nu_0} > 1$), the line saturates and cannot exceed S_ν

Emission line



Absorption line



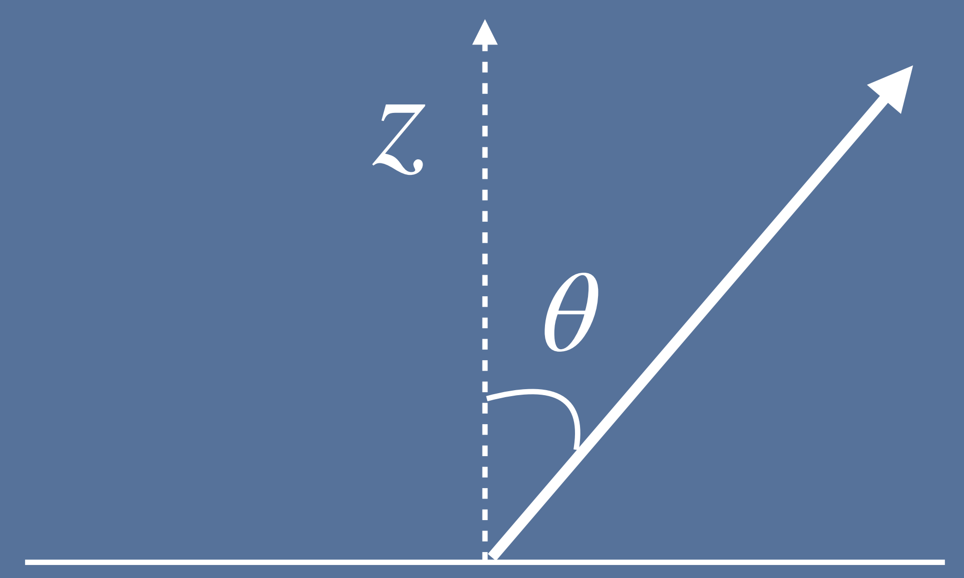
5.2 Plane parallel medium

- The homogeneity hypothesis is often not very realistic (which does not prevent us from using it)
- In certain media, we can consider an axial symmetry if we suppose that the object is made of parallel layers, ie the only variations are in the (vertical) z direction.
- This approximation is very important, in particular to treat stellar and planetary atmospheres
- In this case, the gas variables (like temperature, density) do not depend on x and y, but only on z (the vertical coordinate). The problem has a translation symmetry along x and y, and also a rotation symmetry in the plane (xy).
- This reduces the dimension of the problem from 3 spatial dimensions to one, and from a total of 6 dimensions to 3. The remaining dimensions are the z coordinate, the angle θ such as $\mu = \cos\theta$ and the frequency ν. The angle ϕ disappears because of the rotation symmetry in the plane (xy)

5.2 Plane parallel medium

- However, even though the geometry is formally 1-D, this does not mean that photons move along the z axis. Photons move in the 3 directions of space, and the problem is really a 3-D problem, we just do not have to look at the dependency in x and y. The only dependency that counts for the direction is that on θ.

- Solving the radiative transfer in the plane parallel geometry gives the 3D solution



ds is along the ray

$$dz = \cos\theta\, ds = \mu\, ds$$

- The transfer equation becomes: $$\mu \frac{dI_\nu(z,\mu)}{dz} = j_\nu(z) - \alpha_\nu(z)\, I_\nu(z,\mu)$$

$$\mu \frac{dI_\nu(z,\mu)}{dz} = \alpha_\nu(z)[S_\nu(z) - I_\nu(z,\mu)]$$

5.2 Plane parallel medium

- Integrating the transfer equation along dz is equivalent to integrating the equation with $\frac{dI_\nu}{ds}$ along the direction of propagation $\vec{n}$
- The moments of the intensity in plane parallel geometry can be written

$$J_\nu = \frac{1}{2} \int_{-1}^{1} I_\nu(\mu)\, d\mu$$

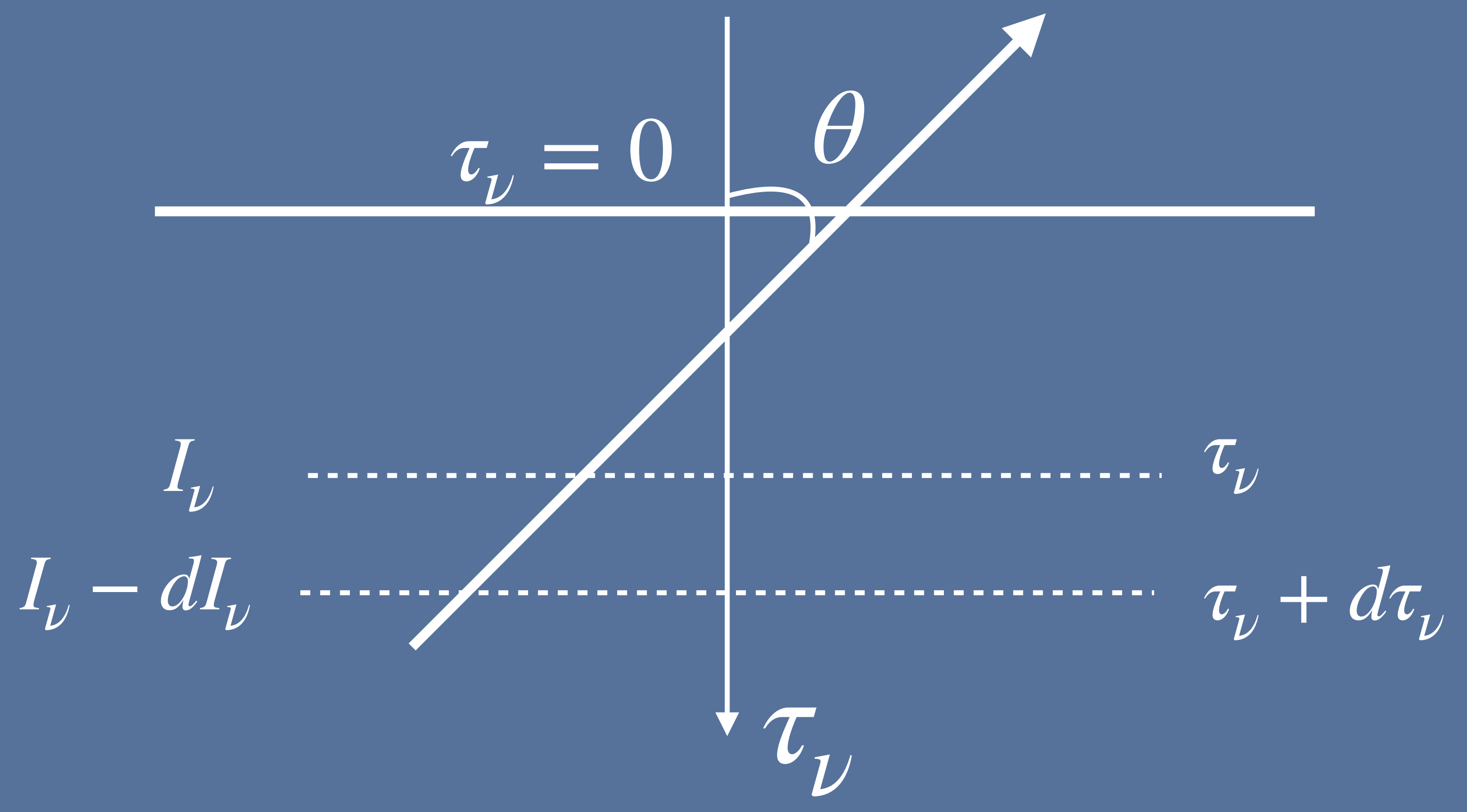
$$H_\nu = \frac{1}{2} \int_{-1}^{1} I_\nu(\mu)\, \mu\, d\mu$$

$$K_\nu = \frac{1}{2} \int_{-1}^{1} I_\nu(\mu)\, \mu^2\, d\mu$$

These quantities are all scalar, because we are only interested in components along z

5.2 Plane parallel medium

- For a stellar atmosphere, we rather use optical depth rather than optical thickness (we are more interested in the observer's point of view)



- With these definitions, the transfer equation is written as follows:

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

with $d\tau_\nu = -\alpha_\nu\, dz = -\alpha_\nu\, \mu\, ds$

- We will now integrate this equation formally: we multiply each term by $\exp(-\tau_\nu/\mu)$ and integrate the left side of the equation by part

5.2 Plane parallel medium

$$\left[I_\nu(\tau'_\nu)\,\exp\left(-\frac{\tau'_\nu}{\mu}\right)\right]_{\tau_{\nu 1}}^{\tau_{\nu 2}} + \int_{\tau_{\nu 1}}^{\tau_{\nu 2}} \frac{1}{\mu}\exp\left(-\frac{\tau'_\nu}{\mu}\right) I_\nu(\tau_\nu{}')\,d\tau_\nu{}' = \int_{\tau_{\nu 1}}^{\tau_{\nu 2}} \frac{1}{\mu}\exp\left(-\frac{\tau'_\nu}{\mu}\right) I_\nu(\tau_\nu{}')\,d\tau_\nu{}' - \int_{\tau_{\nu 1}}^{\tau_{\nu 2}} \frac{1}{\mu}\exp\left(-\frac{\tau'_\nu}{\mu}\right) S_\nu(\tau_\nu{}')\,d\tau_\nu{}'$$

$$\Rightarrow \left[I_\nu(\tau'_\nu)\,\exp\left(-\frac{\tau'_\nu}{\mu}\right)\right]_{\tau_{\nu 1}}^{\tau_{\nu 2}} = -\int_{\tau_{\nu 1}}^{\tau_{\nu 2}} \frac{1}{\mu}\exp\left(-\frac{\tau'_\nu}{\mu}\right) S_\nu(\tau_\nu{}')\,d\tau_\nu{}'$$

Formal solution - to obtain a more explicit solution, we need boundary conditions

5.2.1 Stellar atmosphere

- There are two boundary conditions for a stellar atmosphere
  - There is no incoming radiation at the surface. This means that for $\mu < 0$ (incoming radiation), the intensity at the stellar surface, which is defined by a zero optical depth, is zero:
    $\mu < 0, \quad \mu = -|\mu|, \quad I_\nu(\tau_{\nu_1} = 0) = 0$

    The formal solution becomes $I_\nu(\tau_\nu)\,\exp\left(-\frac{\tau_\nu}{\mu}\right) - I_\nu(0) = -\int_0^{\tau_\nu} \frac{1}{\mu} \exp\left(-\frac{\tau_\nu'}{\mu}\right) S_\nu(\tau_\nu')\, d\tau_\nu'$

    $\forall \quad \mu < 0 \quad I_\nu(0) = 0$

    $$I_\nu^-(\tau_\nu) = \int_0^{\tau_\nu} \frac{S_\nu(\tau_\nu')}{|\mu|} \exp\left(-\frac{\tau_\nu' - \tau_\nu}{\mu}\right) d\tau_\nu'$$

    Note that the intensity for $\mu < 0$ is often written I_ν^-

5.2.1 Stellar atmosphere

- For $\tau_\nu \rightarrow \infty$, the radiation cannot be infinite, so that the first term on the left-hand side of the equation $I_\nu(\tau_\nu \rightarrow \infty)\ \exp\left(-\frac{\tau_\nu \rightarrow \infty}{\mu}\right)$ tends to 0 (the intensity is finite and does not increase exponentially, so the product of the intensity by an exponential that tends towards 0 has to be 0)

$$I_\nu(\tau_\nu \rightarrow \infty)\ \exp\left(-\frac{\tau_\nu \rightarrow \infty}{\mu}\right) - I_\nu(\tau_\nu)\ \exp\left(-\frac{\tau_\nu}{\mu}\right) = -\int_{\tau_\nu}^{\infty} \frac{S_\nu(\tau_\nu{}')}{\mu} \exp\left(-\frac{\tau'_\nu}{\mu}\right) d\tau_\nu{}'$$

$$\mu > 0: \quad I_\nu^+(\tau_\nu) = \int_{\tau_\nu}^{\infty} \frac{S_\nu(\tau_\nu{}')}{\mu} \exp\left(-\frac{\tau'_\nu - \tau_\nu}{\mu}\right) d\tau_\nu{}'$$

The radiation coming out of the surface is then

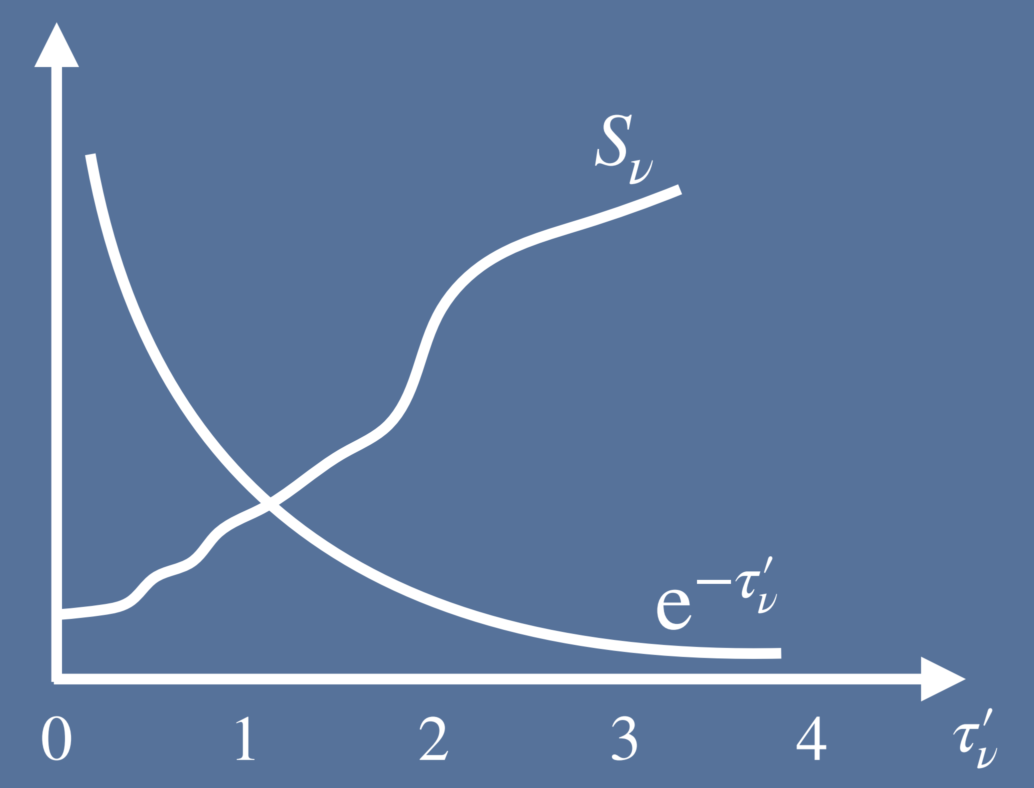
$$I_\nu^+(0) = \int_0^{\infty} \frac{S_\nu(\tau_\nu{}')}{\mu} \exp\left(-\frac{\tau'_\nu}{\mu}\right) d\tau_\nu{}'$$

5.2.1 Stellar atmosphere

- For $\mu = 1$, (vertical direction), we obtain:

$$I_\nu^+(\tau_\nu = 0, \mu = 1) = \int_0^\infty S_\nu(\tau_\nu{}') \exp\left(-\tau_\nu'\right) d\tau_\nu{}'$$

- The outgoing intensity is determined by the source function, with variations towards the inside of the medium damped by a factor $\exp\left(-\tau_\nu'\right)$. This factor quickly decreases with increasing optical depth, and limits the value of the integral to the top layers of the object.



From which altitude does the radiation escape?

5.2.1 Stellar atmosphere

Eddington-Barbier approximation

- We can develop the source function in a Taylor series:

$$S_\nu(\tau_\nu) = \sum_{n=0}^{\infty} a_n \, \tau_\nu^n = a_0 + a_1 \, \tau_\nu + a_2 \, \tau_\nu^2 + \ldots + a_n \, \tau_\nu^n$$

Which we inject into the expression of the intensity

$$I_\nu^+(\tau_\nu = 0, \mu) = \int_0^\infty \frac{S_\nu(\tau_\nu{}')}{\mu} \exp\left(-\frac{\tau_\nu'}{\mu}\right) d\tau_\nu{}' = a_0 + a_1 \, \mu + 2 \, a_2 \, \mu^2 + \ldots + n! \, a_n \, \mu^n$$

Where we have used $\displaystyle\int_0^\infty x^n \, \exp(-x) \, dx = n!$

5.2.1 Stellar atmosphere

Eddington-Barbier approximation

- If we truncate after the first terms:

$$I_\nu^+(\tau_\nu = 0, \mu) = a_0 + a_1\,\mu = S_\nu(\tau_\nu = \mu)$$

- This is the Eddington-Barbier approximation: the observed intensity is approximately equal to the source function where the optical depth is of the order of μ.

- This relation is exact if S_ν varies linearly with with optical depth, but in the general case, it is an approximation

5.2.1 Stellar atmosphere

Eddington-Barbier approximation

- For an outward vertical intensity ($\mu = 1$)

$$I_\nu^+(\tau_\nu = 0, \mu = 1) = S_\nu(\tau_\nu = 1)$$

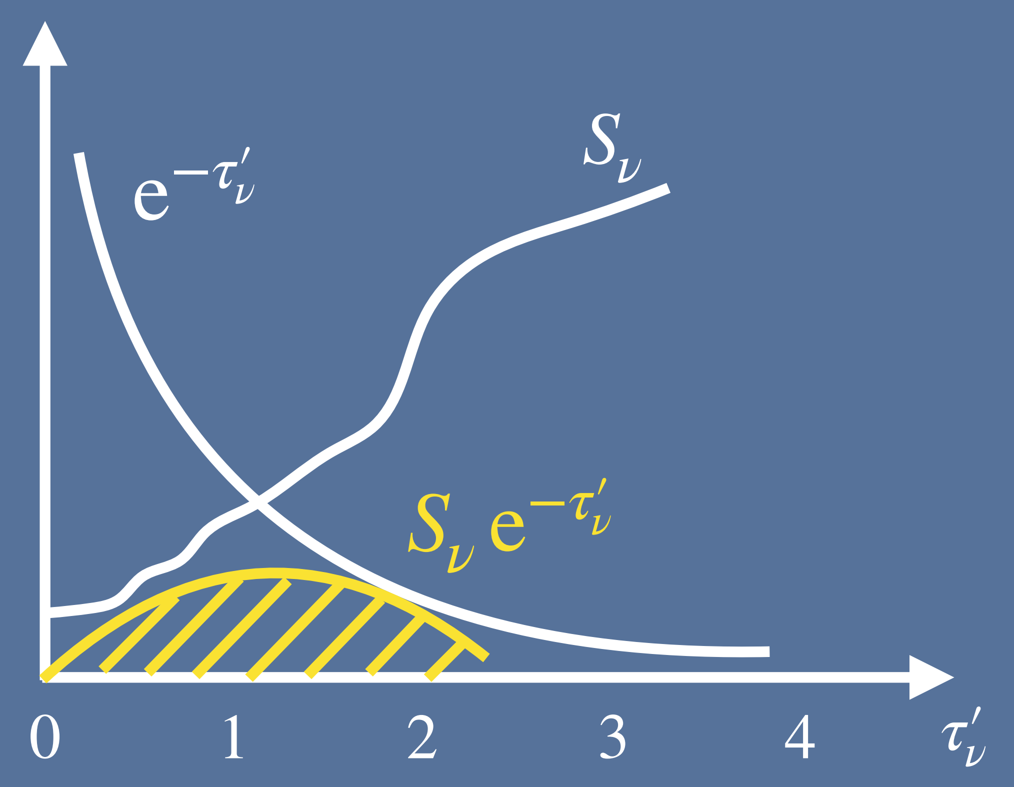
- The emerging intensity is close to the source function at on optical depth of 1 (one mean free path from the surface).

- It is often said that the photons come from an optical depth of one.

- This does not mean that all photons escaped from an optical depth of $\tau_\nu = 1$

- Photons escape from the whole medium but are considered collectively by the value of the source function at $\tau_\nu = 1$

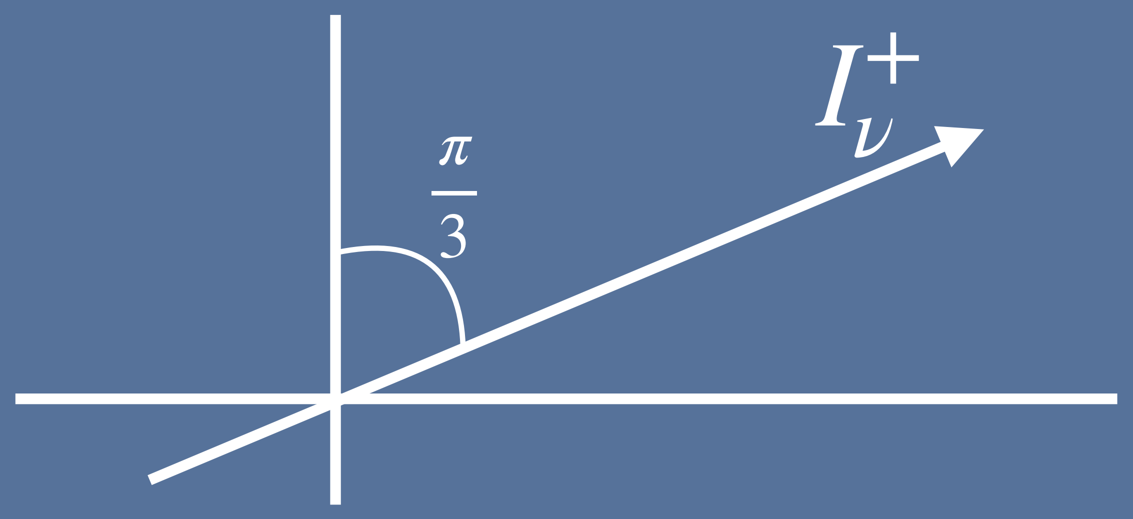
- The integrand $S_\nu(\tau_\nu{}') \exp\left(-\tau_\nu'\right)$ extends from the surface to large τ_ν values (even $\tau_\nu \sim 10$) until the exponentiel factor “cuts” it



5.2.1 Stellar atmosphere

Eddington-Barbier approximation

- If $\mu \neq 1$, the ray is slanted. What matters is the optical depth along the direction of propagation of the ray



If $\theta = \pi/3, \quad \mu = 1/2$

$$I_\nu^+(\tau_\nu = 0, \mu) = S_\nu(\tau_\nu = 1/2)$$

- What is the flux arising from an optically thick medium for which S_ν linearly varies with τ_ν?

- $$F_\nu^+ = F_\nu(\mu > 0) = 2\pi \int_0^1 \mu\, I_\nu\, d\mu = 2\pi \int_0^1 \int_0^\infty \frac{S_\nu(\tau_\nu{}')}{\mu} \exp\left(-\frac{\tau_\nu'}{\mu}\right) \mu\, d\tau_\nu{}'$$

- Assuming $S_\nu(\tau_\nu) = a_0 + a_1\, \tau_\nu \Rightarrow I_\nu^+(0, \mu) = a_0 + a_1\, \mu$

5.2.1 Stellar atmosphere

Eddington-Barbier approximation

$$F_\nu^+(0) = 2\pi \int_0^1 (a_0\mu + a_1\,\mu^2)\, d\mu = 2\pi \left[\frac{a_0\,\mu^2}{2} + \frac{a_1\,\mu^3}{3}\right]_0^1 = \pi \left[a_0 + \frac{2}{3}a_1\right]$$

$$\Rightarrow \quad F_\nu^+(0) = \pi\, S_\nu(\tau_\nu = 2/3)$$

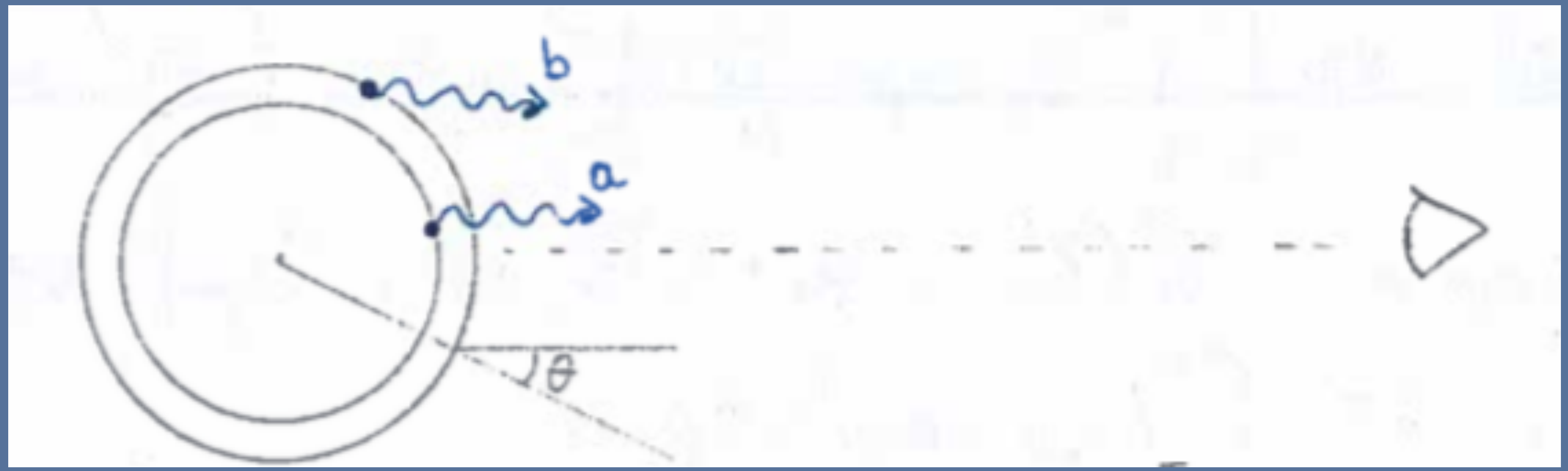
- This is the Eddington Barbier approximation for the flux at the surface. It is an exact relation is the source function varies linearly with τ_ν, and an approximation otherwise

- The outgoing flux and intensity are approximately equal to the source fonction in the superficial layers (e.g. of stars), those where $\tau \leq 1$, and the most internal layers do not contribute to the outgoing radiation, the source function being exponentially absorbed

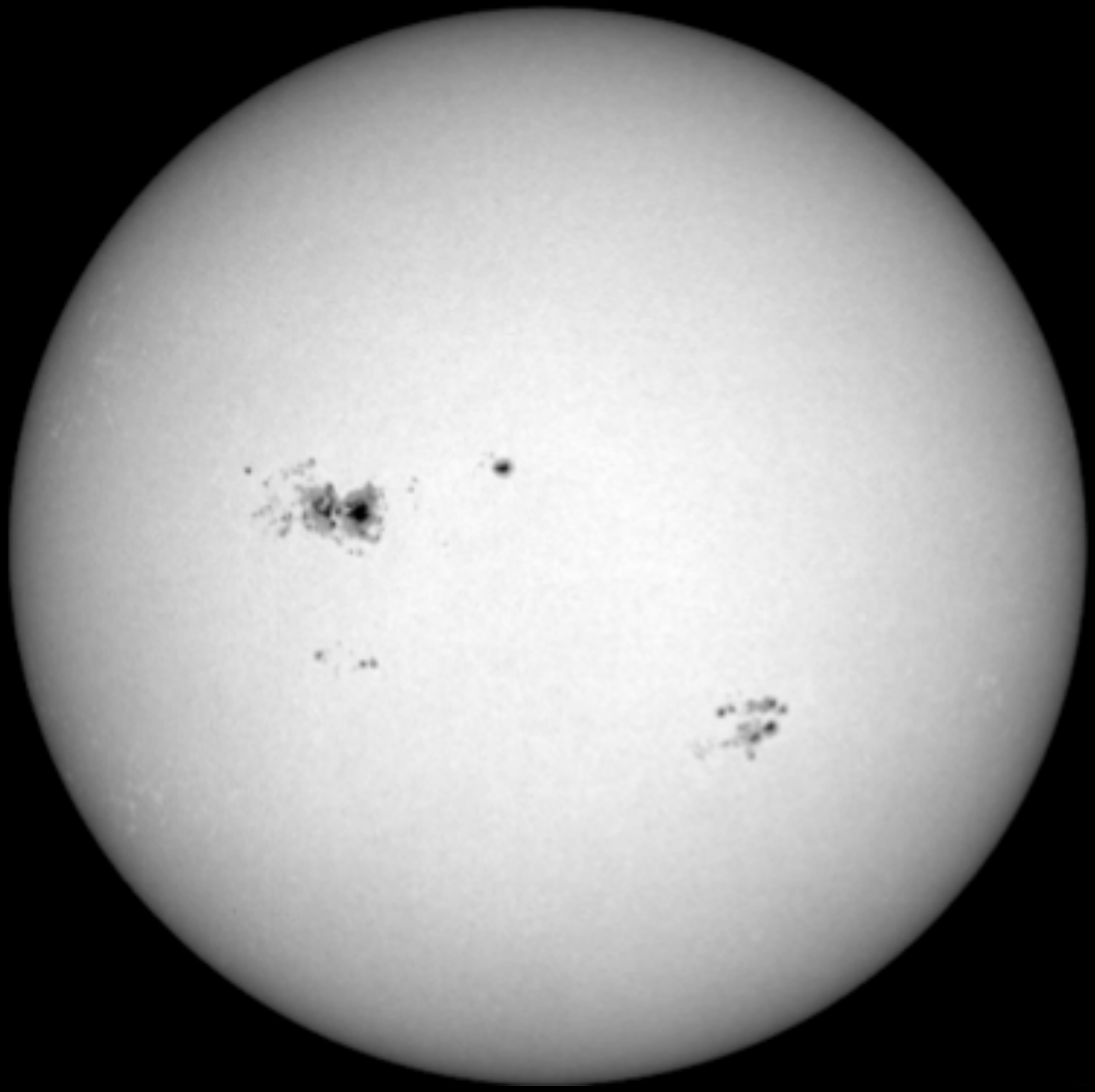
5.2.1 Stellar atmosphere

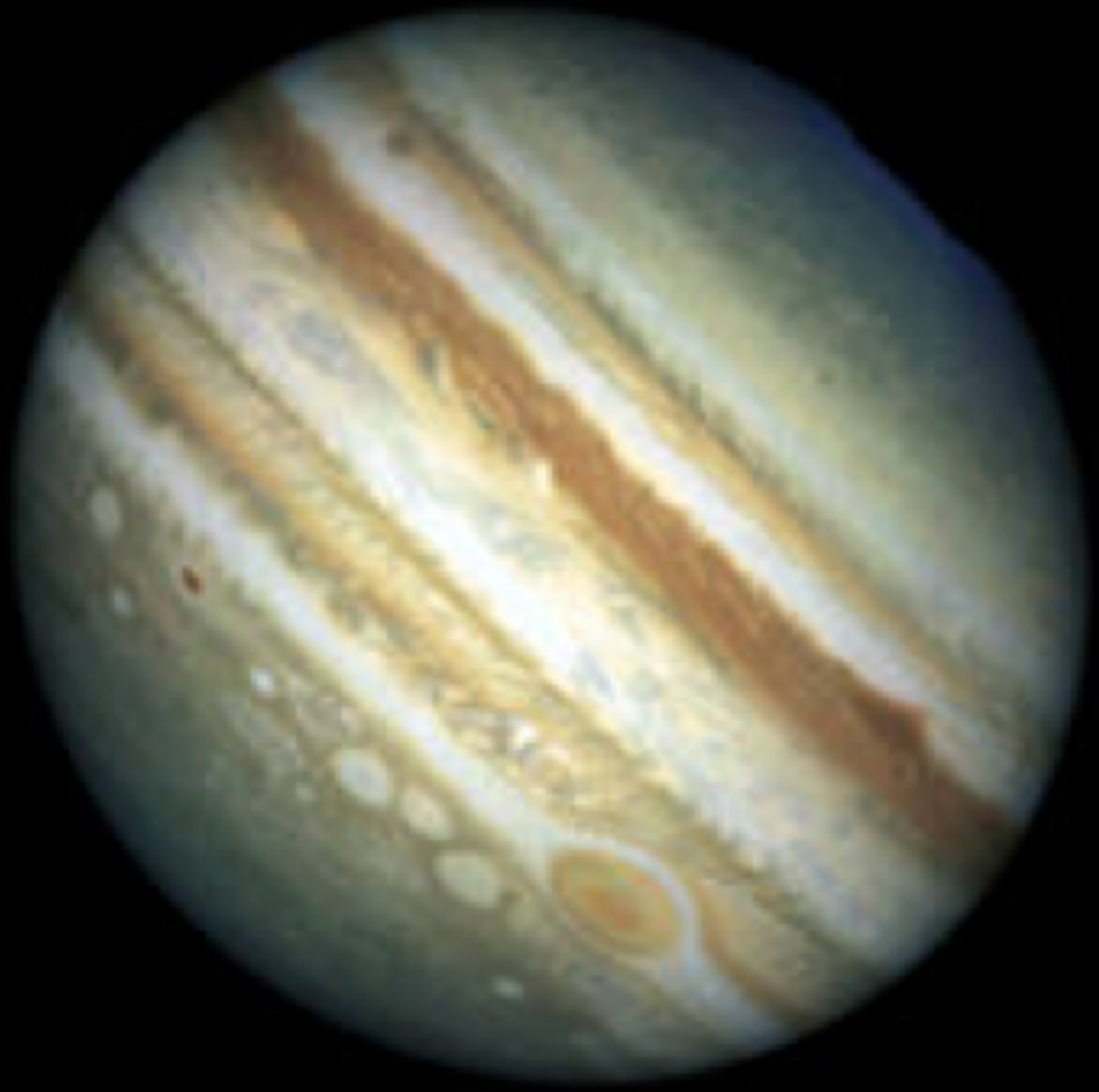
Eddington-Barbier approximation

- In fact, the source function decreases towards the stellar surface
- For the Sun, we can see several points on the solar disk
  - At the edge, we see down to a depth of $\mu = 0$ the source function in the most superficial layer
  - At the centre ($\mu = 1$) we see deeper layers
  - Because S_ν decreases towards the outer layers, the edges will appear less bright than the centre
  - This is called limb darkening



5.2.1 Stellar atmosphere





Limb darkening

5.2.1 Stellar atmosphere

- For an optically thin object, the intensity is

  $I_\nu \simeq S_\nu \tau_\nu = \alpha_\nu S_\nu D$, with D the thickness of the (homogeneous) medium

- For an optically thick object, we have

  $I_\nu \sim S_\nu(\tau_\nu = \mu)$

- In both cases the source function S_ν and the extinction coefficient α_ν have to be specified. In the optically thick case, we must also know α_ν to determine the location where $\tau_\nu = \mu$

- These quantities are different depending on the radiation process

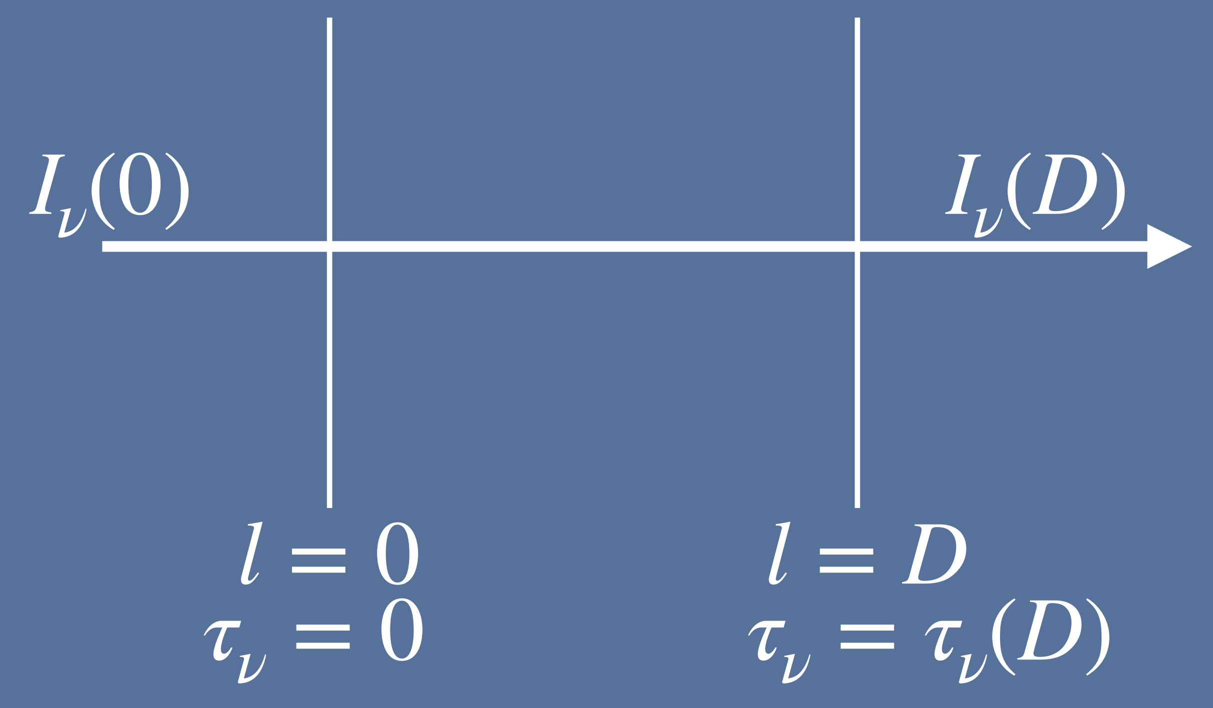
5.2.1 Stellar atmosphere

- If for a narrow frequency range (ie a spectral line), the absorption coefficient is much larger than for the neighbouring frequencies, the outgoing intensity will come from superficial layers where the source function is smaller and from deeper layers where the source function is larger in the neighbouring frequencies
- The calculation of the integral $I_\nu^+(\tau_\nu = 0, \mu) = \int_0^\infty \frac{S_\nu(\tau_\nu{}')}{\mu} \exp\left(-\frac{\tau_\nu'}{\mu}\right) d\tau_\nu{}'$ is one of the most difficult tasks in astrophysics
- Indeed, it consists in determining for each height in the stellar atmosphere both the optical depth (which depends on that of all above-located points) and the value of the source function, which depends on the temperature, itself a function of the way the radiation varies throughout the atmosphere
- Moreover the various frequencies are coupled and the intensities varies very rapidly as a function of frequencies in the spectral lines

5.2.2 Homogeneous finite plan parallel layer

- This is approximation is often used for tenuous media like ionised nebula of interstellar clouds

- We have already seen this in §5.1 in the general case and in the case where $\mu \neq 0$



- Boundary conditions
  - $I_\nu(\tau_\nu = 0) = I_{\nu,0}$ incoming intensity in the layer at $\tau_\nu = 0$
  - No incoming radiation for $\tau_\nu = \tau_\nu(D)$

5.2.2 Homogeneous finite plan parallel layer

- The solution of the radiative transfer equation is

$$I_\nu(\tau_\nu(D), \mu) = I_{\nu,0} \exp\left(-\frac{\tau_\nu(D)}{\mu}\right) + \int_0^{\tau_\nu(D)} S_\nu \exp\left(-\frac{\tau'_\nu}{\mu}\right) \frac{d\tau'_\nu}{\mu}$$

$$= I_{\nu,0} \exp\left(-\frac{\tau_\nu(D)}{\mu}\right) + S_\nu \left[1 - \exp\left(-\frac{\tau_\nu(D)}{\mu}\right)\right]$$

(S_ν is the source function and is constant in the layer)

- Non emitting case: $I_\nu(\tau_\nu(D), \mu) = I_{\nu,0} \exp\left(-\frac{\tau_\nu(D)}{\mu}\right)$

  e.g. a cold cloud in front of a bright source. The intensity is equal to the incoming intensity attenuated by the absorption in the layer

5.2.2 Homogeneous finite plan parallel layer

- Optically thin case: $\tau_\nu \ll 1$

$$I_\nu(\tau_\nu(D), \mu) = I_{\nu,0} \exp\left(-\frac{\tau_\nu(D)}{\mu}\right) + S_\nu \frac{\tau_\nu(D)}{\mu}$$

$$= I_{\nu,0} \exp\left(-\frac{\tau_\nu(D)}{\mu}\right) + \frac{D\, j_\nu}{\mu}$$

The outgoing intensity is equal to the integrated emissivity, increased by the incoming intensity attenuated by the layer absorption

- Optically thick layer: $\tau_\nu \gg 1$

$I_\nu(\tau_\nu(D), \mu) \simeq S_\nu$: The outgoing intensity is equal to the source function

5.2.2 Homogeneous finite plan parallel layer

- The resulting fluxes are
  - Optically thin case: $F_\nu = 2\pi D j_\nu$ $\quad\left(= 2\pi \int_0^1 \frac{D j_\nu}{\mu} \mu \, d\mu \right)$
  - Optically thick case: $F_\nu = \pi S_\nu$
- And the luminosity $L_\nu = F_\nu \times \text{surface}$
  - Optically thin case: $L_\nu = 2\pi j_\nu \times \text{volume}$
  - Optically thick case: $L_\nu = \pi S_\nu \times \text{surface}$
- For an optically thin layer, we are sensitive to the whole emissivity and the power is proportional to the volume, whereas for an optically thick layer, we can see the source function, and the power is proportional to the surface

5.3 Eddington approximation

- We consider a plane parallel medium. It is not a necessary condition, but this will help to introduce the method
- This approximation is very much used in the deep layers of stellar atmospheres where plan parallel geometry applies. In this case, we have $\mu = \cos\theta,\, d\mu = \sin\theta\, d\theta$
- In this section, we also consider there is no scattering
- The moments of the intensity can be written:

$$J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu\, d\mu \quad F_\nu = 2\pi \int_{-1}^{1} I_\nu\, \mu\, d\mu \quad P_\nu = \frac{2\pi}{c}\int_{-1}^{1} I_\nu\, \mu^2\, d\mu$$

(we could also use $H_\nu = F_\nu/4\pi$ and $K_\nu = c/4\pi\, P_\nu$)

5.3 Eddington approximation

- The transfer equation is: $\mu \dfrac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$

- Assuming S_ν is isotropic and integrating the transfer equation over $d\Omega$, we obtain: $\dfrac{dF_\nu}{d\tau_\nu} = 4\pi\,(J_\nu - S_\nu)$,

  where we have defined $d\tau_\nu = \alpha_\nu\, ds$

- Note that assuming the medium isotropic does not imply that the intensity is isotropic, only the source function is isotropic

- Multiplying the transfer equation by μ, we obtain: $\mu^2 \dfrac{dI_\nu}{d\tau_\nu} = \mu\,(I_\nu - S_\nu)$

- After integration over $d\Omega$: $c\dfrac{dP_\nu}{d\tau_\nu} = F_\nu$

  The term $\int \mu\, S_\nu\, d\Omega$ disappears because S_ν is isotropic

5.3 Eddington approximation

- From these two “moments of the transfer equation”, we derive
$$c\frac{d^2P_\nu}{d\tau_\nu^2} = 4\pi\,(J_\nu - S_\nu)$$

- The Eddington approximation consists in adding a closure relation: $P_\nu = \dfrac{4\pi}{3c}J_\nu$

- This equation is exact in an isotropic medium: see Chapter 2, $u_\nu = \dfrac{4\pi}{c}J_\nu$ (always true) and $P_\nu = \dfrac{u_\nu}{3}$ (valid in an isotropic medium)

- The approximation consists in using this method when the medium is nearly isotropic, like in the deep layers of stellar atmospheres. It is often used and gives good results

5.3 Eddington approximation

- Combining these last two equations, we obtain an equation for the mean intensity, which only depends on the direction μ

$$\frac{1}{3}\frac{d^2 J_\nu}{d\tau_\nu^2} = J_\nu - S_\nu$$

- This is the Eddington equation. It remains very hard to integrate in the general case where S_ν depends on J_ν (typically when there is scattering) and when frequencies are coupled.

- To solve this equation in simple cases, we can use two slightly different approximations for the intensity. Both these approximations lead to

$$P_\nu = \frac{4\pi}{3c} J_\nu$$

5.3 Eddington approximation

1. Semi-isotropy approximation for the radiation: the radiation is assumed to be isotropic in each of both hemispheres $\mu > 0$ and $\mu < 0$: $I_\nu(\mu < 0) = I_\nu^- = \mathrm{cst}$ and $I_\nu(\mu > 0) = I_\nu^+ = \mathrm{cst}$ (another constant)

$$J_\nu = \frac{I_\nu^+ + I_\nu^-}{2} \quad F_\nu = \pi (I_\nu^+ - I_\nu^-) \quad P_\nu = \frac{2\pi}{3c}(I_\nu^+ + I_\nu^-) = \frac{4\pi}{3c} J_\nu$$

2. Two-stream approximation: the intensity is assumed to be confined to two directions, for which the angle cosines are $1/\sqrt{3}$ for I_ν^+ and $-1/\sqrt{3}$ for I_ν^-.

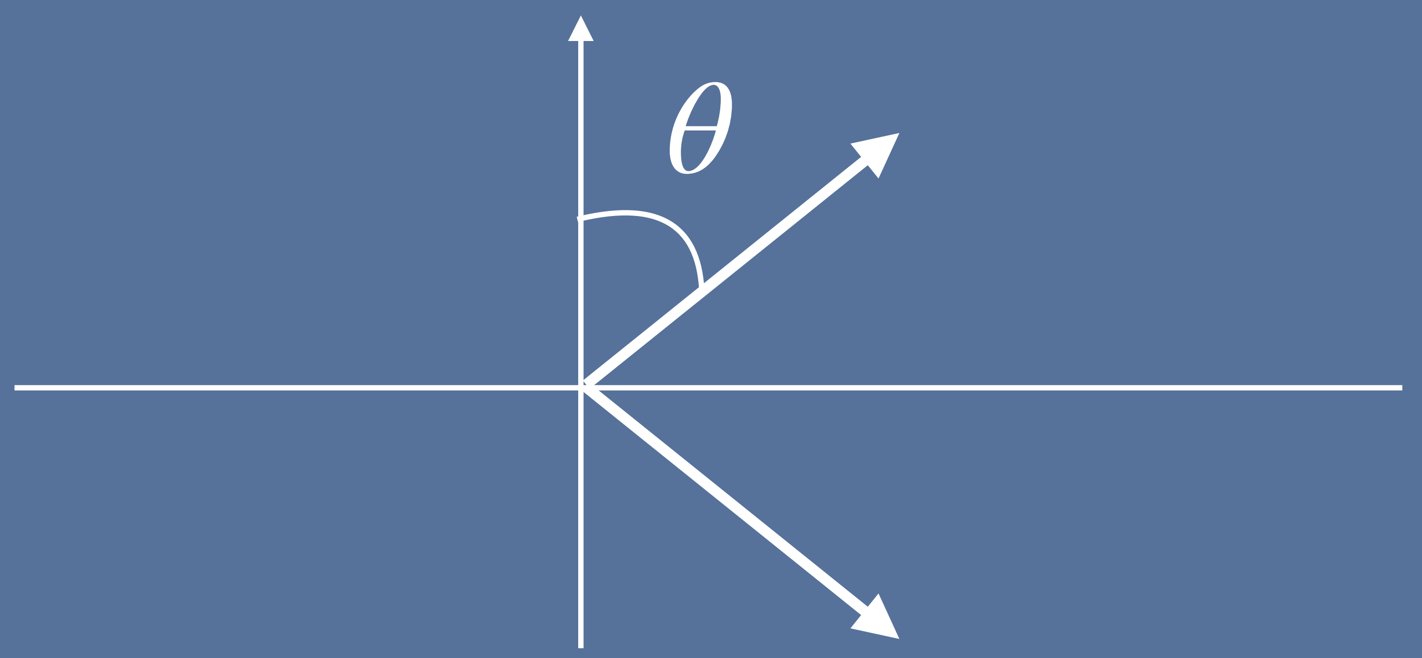
$$J_\nu = \frac{I_\nu^+ + I_\nu^-}{2} \quad F_\nu = \frac{2\pi}{\sqrt{3}} (I_\nu^+ - I_\nu^-) \quad P_\nu = \frac{2\pi}{3c}(I_\nu^+ + I_\nu^-) = \frac{4\pi}{3c} J_\nu$$

F_ν is slightly different for those two approximations

5.3 Eddington approximation

Solution for the Eddington equation in the case of a semi-infinite layer (stellar atmosphere) and

$$\cos\theta = \pm\frac{1}{\sqrt{3}}$$



We assume in addition that S_ν varies linearly with τ_ν

The general solution of the equation $\frac{1}{3}\frac{d^2 J_\nu}{d\tau_\nu^2} = J_\nu - S_\nu$ is:

$$J_\nu - S_\nu = C_1 \exp(\sqrt{3}\,\tau_\nu) + C_2 \exp(-\sqrt{3}\,\tau_\nu)$$

5.3 Eddington approximation

To determine the constants C_1 and C_2, we use the boundary conditions

- for $\tau_\nu \to \infty$, $J_\nu - S_\nu$ remains finite, therefore $C_1 = 0$

- for $\tau_\nu = 0$, at the surface, there is no incoming intensity, ie $I_\nu^- = 0$

Using the two-stream approximation, we have to solve

$$J_\nu = \frac{I_\nu^+ + I_\nu^-}{2} \quad \text{and} \quad F_\nu = \frac{2\pi}{\sqrt{3}}(I_\nu^+ - I_\nu^-) \;\Rightarrow\; I_\nu^+ \;\text{ and }\; I_\nu^-$$

$$I_\nu^+ = J_\nu + \frac{\sqrt{3}}{4\pi} F_\nu \qquad I_\nu^- = J_\nu - \frac{\sqrt{3}}{4\pi} F_\nu$$

5.3 Eddington approximation

With $c\dfrac{dP_\nu}{d\tau_\nu} = F_\nu$ and $P_\nu = \dfrac{4\pi}{3c} J_\nu$, we obtain $\dfrac{4\pi}{3}\dfrac{dJ_\nu}{d\tau_\nu} = F_\nu$, which we insert in the previous expressions for I_ν^+ and I_ν^- :

$$I_\nu^+ = J_\nu + \frac{1}{\sqrt{3}}\frac{dJ_\nu}{d\tau_\nu} \qquad I_\nu^- = J_\nu - \frac{1}{\sqrt{3}}\frac{dJ_\nu}{d\tau_\nu}$$

Using the boundary conditions : $J_\nu(0) = \dfrac{1}{\sqrt{3}}\dfrac{dJ_\nu}{d\tau_\nu}\Bigg|_{\tau_\nu=0}$

And $\quad J_\nu(0) = C_2 + S_\nu(0) = \dfrac{1}{\sqrt{3}}\left(\dfrac{dS_\nu}{d\tau_\nu} - \sqrt{3}\,C_2\right) \quad \Rightarrow \quad C_2$

5.3 Eddington approximation

The solution is therefore

$$J_\nu(\tau_\nu) = S_\nu(\tau_\nu) + \frac{1}{2}\left(\frac{1}{\sqrt{3}}\frac{dS_\nu}{d\tau_\nu} - S_\nu(0)\right)\exp(-\sqrt{3}\tau_\nu)$$

Which verifies $J_\nu = S_\nu$ for $\tau_\nu \rightarrow \infty$.

The flux at the surface can be written

$$F_\nu(0) = \left(S_\nu(0) + \frac{1}{\sqrt{3}}\frac{dS_\nu}{d\tau_\nu}\right)\frac{2\pi}{\sqrt{3}}$$