Radiative Transfer

4. Matter and radiation in equilibrium

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Introduction

- In this chapter, we take a look at thermodynamic equilibrium
- The laws of thermal equilibrium are important, even when the medium is not in thermal equilibrium
- Photons (bosons) follow the Bose-Einstein statistics and can occupy the same quantum state
- Particules (fermions) follow the Fermi-Dirac statistics and obey the Pauli exclusion principle
- Thermal equilibrium is the state of a collection of interacting photons and particules in an enclosure. The energy distribution of photons and particles is at steady state and corresponds to micro-reversibility
- Macroscopically, all equipartition laws are valid and the phenomena are characterised by a single temperature
- Equipartition laws for radiation: Planck, Wien, Kirchhoff, Stefan-Boltzmann
- Equipartition laws for matter: Boltzmann, Maxwell, Saha

1. Radiation at thermodynamic equilibrium

1.1 Planck law

1.1.1 Expression

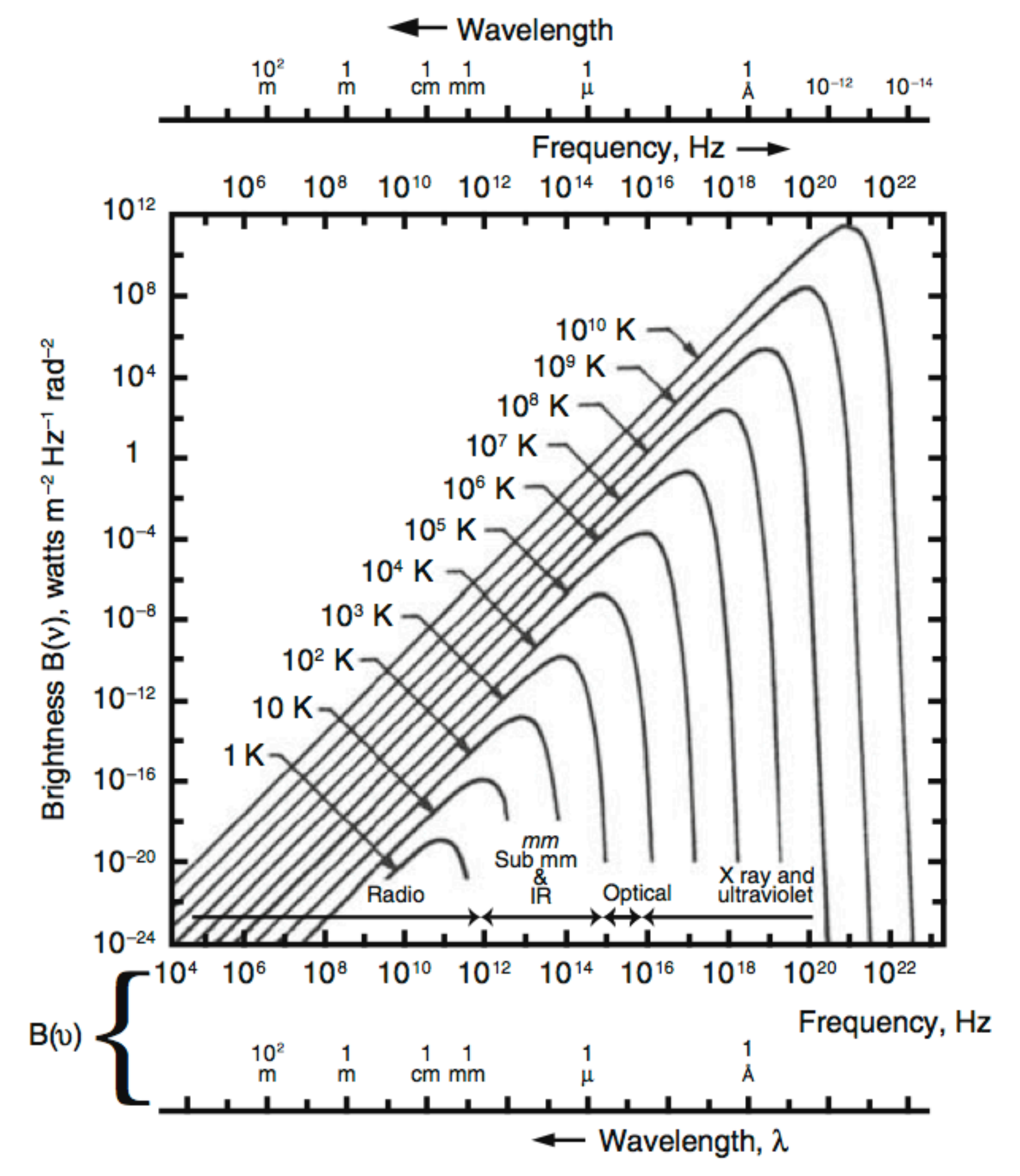
- Energy distribution of a blackbody, of specific intensity B_ν

$$I_\nu = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$$

- unit: $erg\ s^{-1}\ cm^{-2}\ Hz^{-1}\ sr^{-1}$ (cgs) // $W\ m^{-2}\ Hz^{-1}\ sr^{-1}$ (MKSA)
- $h = 6.626\,10^{-27}$ erg s : Planck constant
- $k = 1.38\,10^{-16}$ erg K^{-1}: Boltzmann constant
- T: temperature of the medium

1.1.1 Expression

The Planck function depends only on one parameter, T



1.1.1 Expression

- Planck distribution per wavelength interval

$$B_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k T}\right) - 1}$$

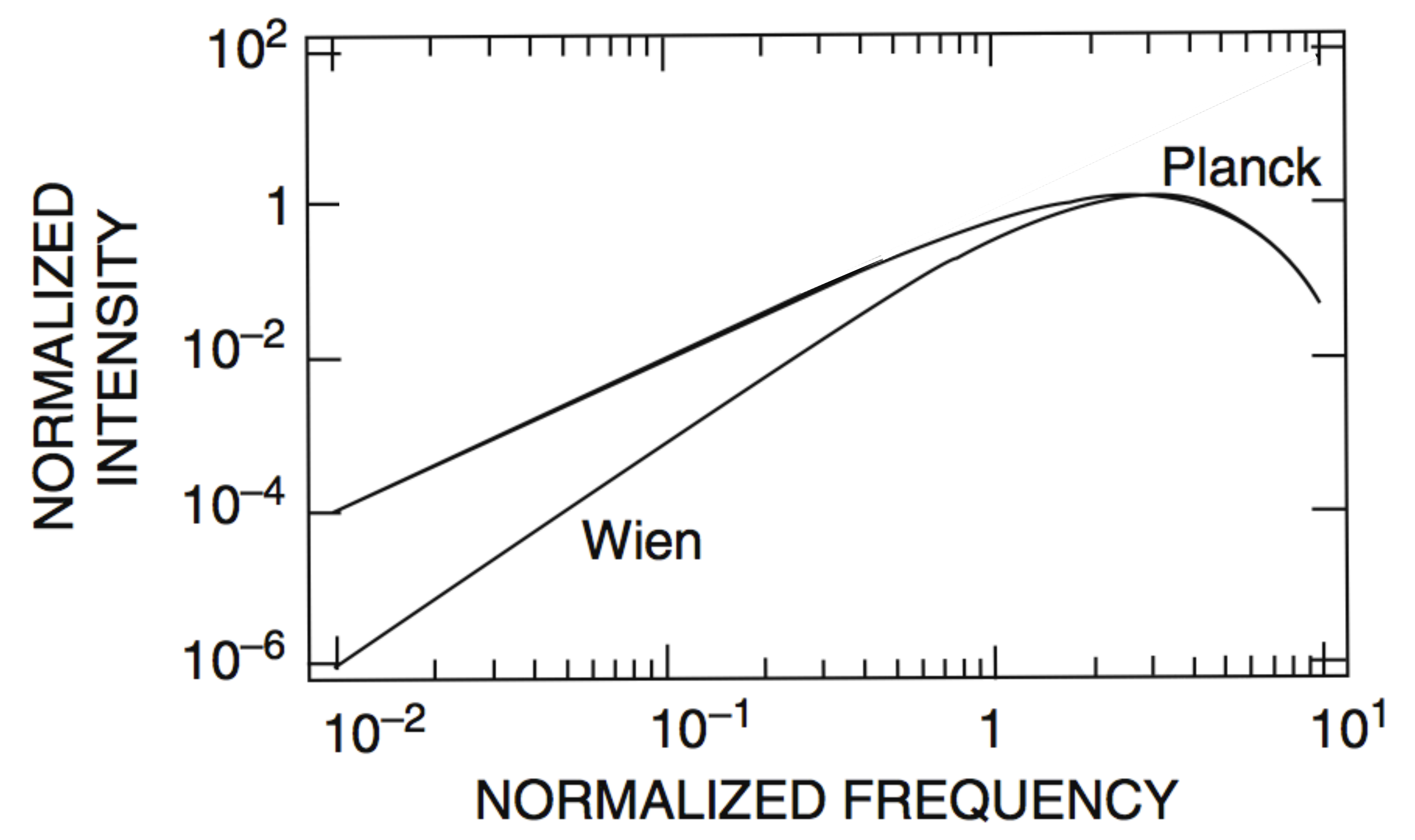
- unit: $\mathrm{erg\ s^{-1}\ cm^{-2}\ cm^{-1}\ sr^{-1}}$ (cgs) // $\mathrm{W\ m^{-2}\ m^{-1}\ sr^{-1}}$ (MKSA)
- What is the relation between B_λ and B_ν ?

1.1.2 Wien approximation

- If ν/T is large enough, i.e. $h\nu \gg kT$, Planck's law can be approximated with

$$B_\nu \simeq \frac{2h\nu^3}{c^2} \exp\left(-\frac{h\nu}{kT}\right)$$

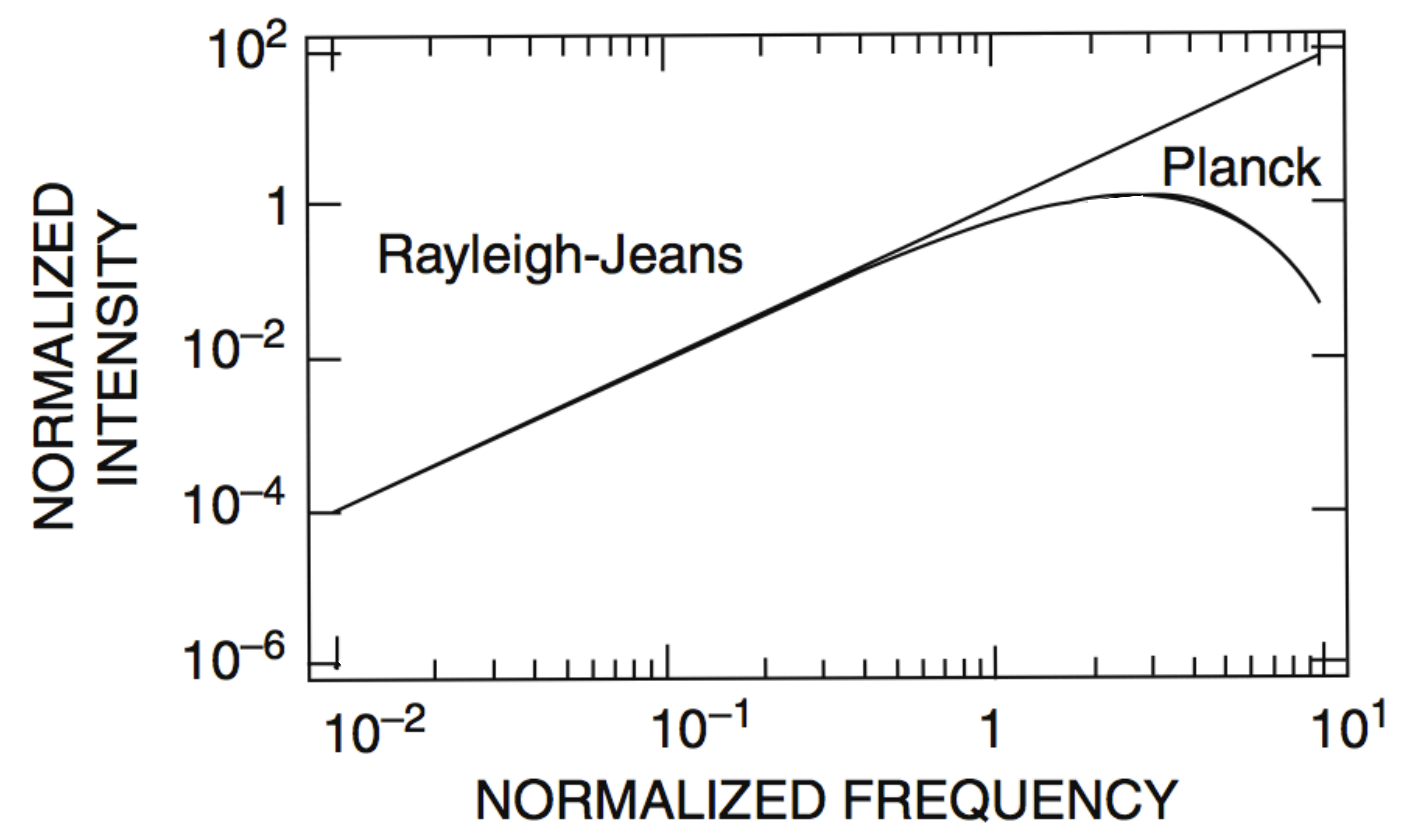
- This is the Wien approximation
- It is used in the X-ray domain
- What is the Wien approximation expressed with B_λ?



1.1.3 Rayleigh-Jeans approximation

- If ν/T is small enough, i.e. $h\nu \ll kT$, $\exp\left(\frac{h\nu}{kT}\right) - 1 \approx \frac{h\nu}{kT}$ Planck's law simplifies into

$$B_\nu \simeq \frac{2\nu^2}{c^2}\, k\, T$$



- This is the Rayleigh-Jeans approximation
- In the RJ approximation, the Planck function is linear with the temperature
- Widely used in radioastronomy
- What is the RJ approximation expressed with B_λ?

1.1.4 Wien’s displacement law

- Where is the maximum intensity of the Planck curve?
- Maximum intensity per frequency interval:

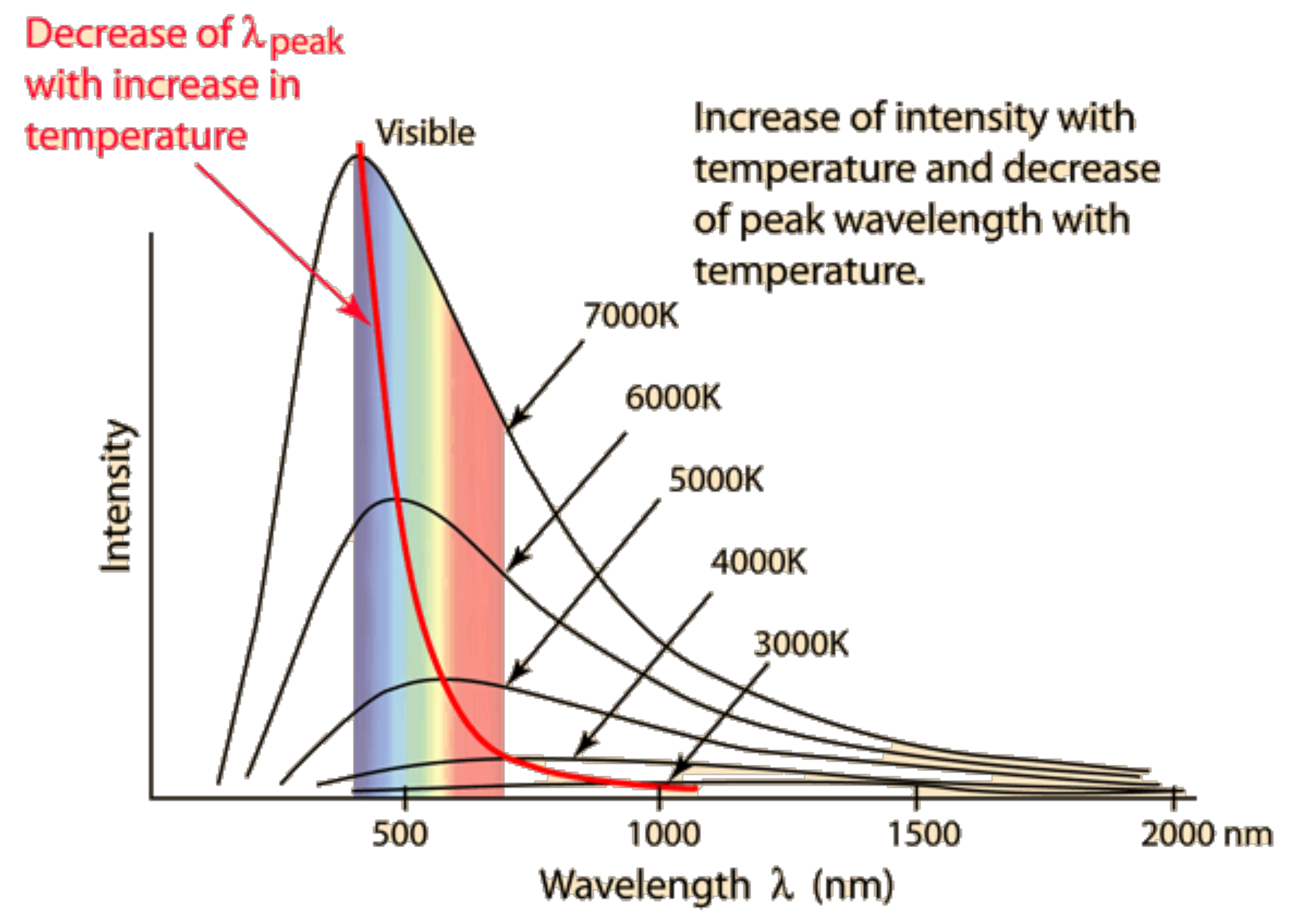
$$\frac{dB_\nu}{d\nu} = 0 \Rightarrow \frac{\nu_{\max}}{T} = 5.88\,10^{10}\ \mathrm{Hz\ K^{-1}}$$

- Maximum intensity per wavelength interval:

$$\frac{dB_\lambda}{d\lambda} = 0 \Rightarrow \lambda_{\max}\,T = 0.29\ \mathrm{cm\ K}$$

- The maximum moves towards shorter wavelengths when the temperature increases
- Are the maxima of B_ν and B_λ at the same frequency / wavelength? Why?

1.1.4 Wien’s displacement law



1.1.5 Stefan-Boltzmann law

- Integrating Planck's law over the entire frequency spectrum one obtains

$$B = \int_0^\infty B_\nu \, d\nu = \frac{\sigma \, T^4}{\pi}$$

- σ is Stefan's constant (not to be confused with the extinction cross section σ_ν, or the scattering coefficient)
- $\sigma = \dfrac{2\pi^5 k^4}{15 \, h^3 c^2} = 5.67 \, 10^{-5} \, \mathrm{erg \, cm^{-2} \, K^{-4} \, s^{-1}}$
- Note that this expression has the dimension of an intensity.
- What is the outwards flux for an isotropically radiating black surface?

1.2 Kirchhoff's law

- In a homogeneous, isothermal, isotropic medium (for instance a medium in an isothermal enclosure), thermodynamic equilibrium applies
- For each beam (in each direction), each frequency and each location in space, the intensity is $I_\nu = B_\nu(T)$
- For the transfer equation: $\dfrac{dI_\nu}{ds} = 0$
- Therefore $j_\nu(s) - \alpha_\nu(s)\, I_\nu(\overrightarrow{n}, s) = j_\nu(s) - \alpha_\nu(s)\, B_\nu(T) = 0$
- $\Rightarrow B_\nu(T) = \dfrac{j_\nu}{\alpha_\nu}$
- This is Kirchhoff's law for radiation in thermodynamic equilibrium: a medium in thermodynamic equilibrium can have any emissivity j_ν and extinction α_ν as long as their ratio is the Planck function.
- We also have $I_\nu(\overrightarrow{n}, s) = \dfrac{j_\nu}{\alpha_\nu}$

1.2 Kirchhoff’s law

- Full thermodynamic equilibrium is not a very interesting case in radiative transfer, why?
- Kirchhoff’s law also applies when the medium is in local thermodynamic equilibrium (LTE): the temperature can vary but in each location we have $B_\nu(T_{\rm loc}) = \dfrac{j_\nu}{\alpha_\nu}$, and the source function is $S_\nu = B_\nu(T_{\rm loc})$.
- At LTE, only the source function S_ν, and not the intensity I_ν, is given by Planck’s law.
- LTE is far from being always valid, but Kirchhoff’s law allows us to greatly simplify the radiative transfer problem

1.2 Kirchhoff's law

- When LTE applies, we can write $\frac{I_\nu(\overrightarrow{n}, s)}{ds} = \alpha_\nu \left[B_\nu(T) - I_\nu(\overrightarrow{n}, s)\right]$
- Planck's function can vary along the beam. The intensity I_ν asymptotically tends towards $B_\nu(T)$.
- If the temperature varies along the beam, the intensity will always tend to approach the Planck function, but with a delay of a few mean free paths.
- At LTE, thermodynamic equilibrium laws for both particles and radiation apply. As for full TE, LTE obeys also microreversibility.
- In order for LTE to apply, the distribution of particles on energy levels has to be governed by collisions, i.e. the density has to be high
  - LTE is valid in the deep layers of stellar atmospheres where density is high
  - In diluted media like most of the interstellar medium, LTE is often far from being valid

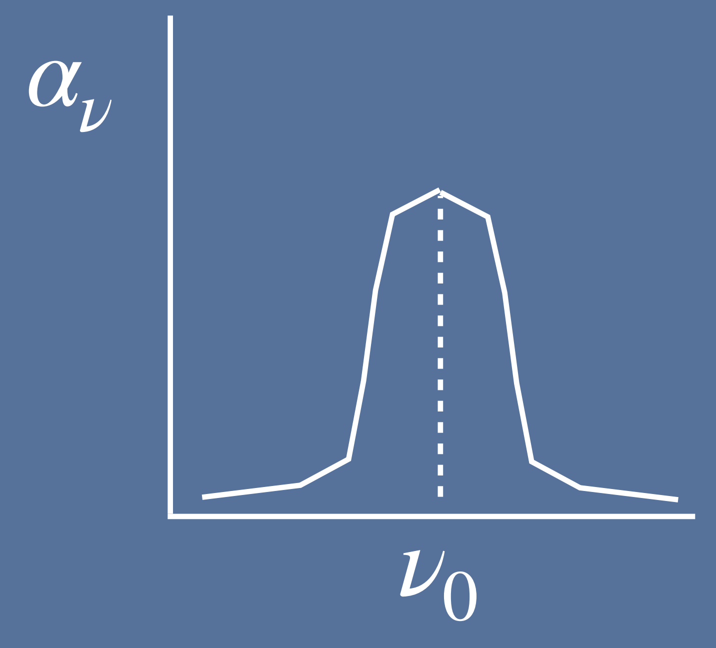
1.2 Kirchhoff’s law

Application: spectral line formation in a stellar atmosphere

- Optically thin case
  - Extinction can be ignored so that the RT equation becomes: $\dfrac{dI_\nu(s)}{ds} = j_\nu(s)$
  - This can be integrated between two coordinates s_0 and s_1: $I_\nu(s_1) = I_\nu(s_0) + \int_{s_0}^{s_1} j_\nu(s)\, ds$
  - If j_ν is large around a frequency ν_0 (spectral line), the function $I_\nu(s_1)$ will also be large around ν_0. If the background intensity $I_\nu(s_0)$ is much larger, the contrast will be small. Most of the time in optically thin sources, the background intensity is small and the spectral line has the same shape as the emissivity function j_ν.
  - Bottom line: independently of the background intensity $I_\nu(s_0)$, the line is seen in emission in the optically thin case

1.2 Kirchhoff's law

Application: spectral line formation in a stellar atmosphere



- "Optically thick" case
  - We can assume additionally that LTE is valid ⇒ Kirchhoff's law applies
  - The medium is made of an optically thick background layer at $T_{\rm bg}$ and a foreground layer at $T_{\rm fg}$, to simulate a temperature gradient
  - The extinction coefficient α_ν is a Gaussian of width γ centred on ν_0
  - D is the geometric thickness of the foreground layer, so that the optical depth of the foreground layer is

    $$\tau_\nu = \alpha_\nu D \text{ and } \tau_\nu = \tau_0 \exp\left(-\frac{(\nu-\nu_0)^2}{\gamma^2}\right)$$

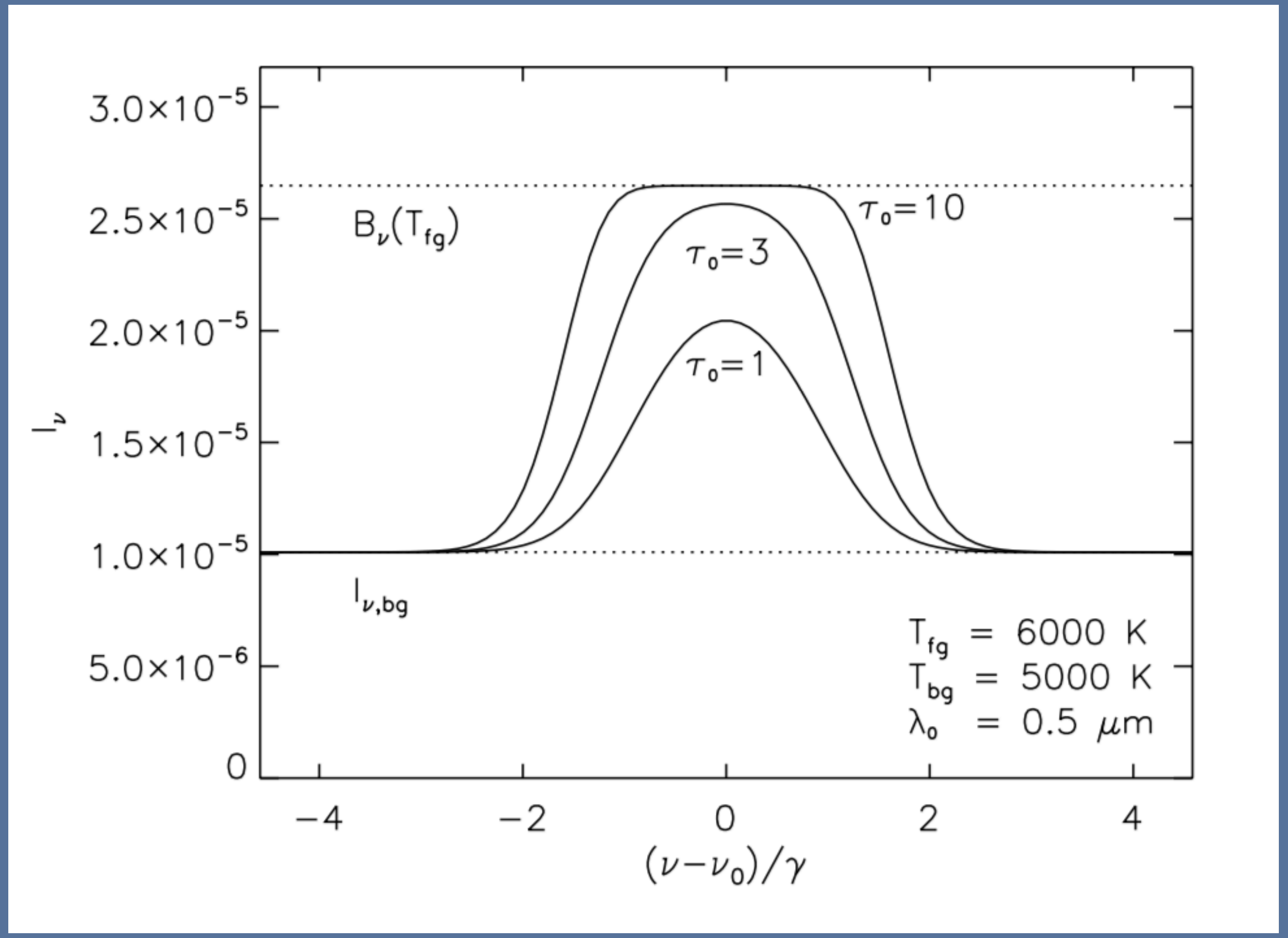
  - The emission of background layer can be approximated with a blackbody function $I_{\nu,\rm bg} = B_\nu(T_{\rm bg})$
  - What does the observed intensity look like?

1.2 Kirchhoff's law

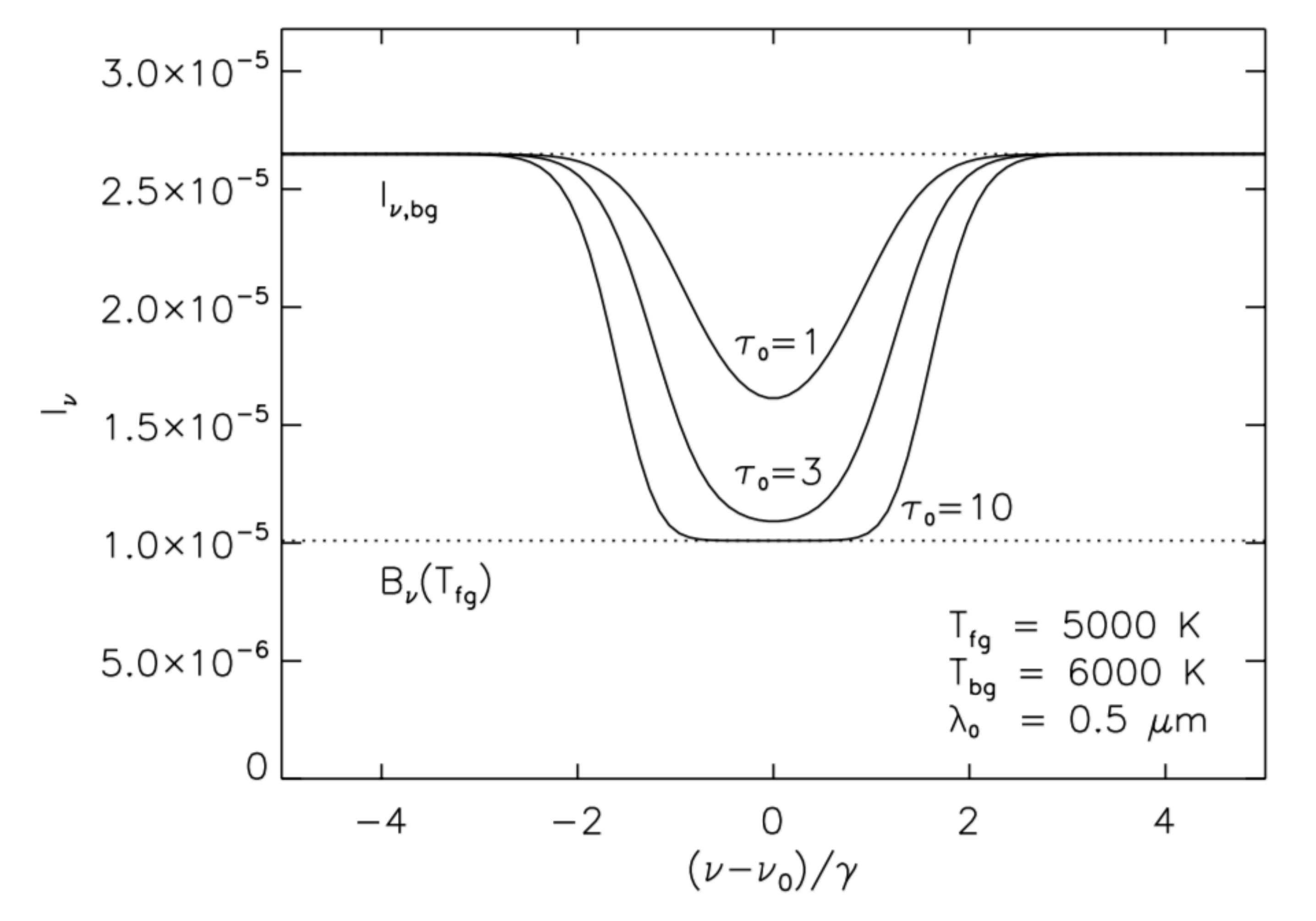
Application: spectral line formation in a stellar atmosphere

- Integration of the RT equation: $I_{\nu,\mathrm{obs}} = I_{\nu,\mathrm{bg}}\, \mathrm{e}^{-\tau_\nu} + (1 - \mathrm{e}^{-\tau_\nu})\, B_\nu(T_{\mathrm{fg}})$
- If we consider 3 optical depths ($\tau_0 = 1,\ \tau_0 = 3,\ \tau_0 = 10$), $\lambda = 0.5\,\mu\mathrm{m}$ ($\nu_0 = 6\ 10^{14}$ Hz) and 2 temperature cases

$T_{\mathrm{bg}} = 5000\,\mathrm{K},\ \mathrm{T_{fg}} = 6000\,\mathrm{K}$



$T_{\mathrm{bg}} = 6000\,\mathrm{K},\ \mathrm{T_{fg}} = 5000\,\mathrm{K}$



1.2 Kirchhoff’s law

Application: spectral line formation in a stellar atmosphere

- Hot layer in front of cold layer ⇒ emission line
- Cold layer in front of hot layer ⇒ absorption line
- For an optically thick atmosphere with a constant temperature ⇒ no line is observed
- The shape of the line is the same as that of the extinction coefficient when $\tau_0 < 1$.
- For $\tau_0 \gg 1$ the line becomes optically thick and saturates.
- This is again the attractor effect: the intensity approaches the Planck function of the foreground layer. When it reaches it, it remains constant
- In the non-LTE case, instead of the Planck function $B_\nu(T_{\rm fg})$, we have the source function $S_{\nu,\rm fg}$ (the rest remains unchanged). It is then possible to have a line even if the temperature is constant, as long as $S_{\nu,\rm fg} \neq I_{\nu,\rm bg}$

2. Matter at thermodynamic equilibrium

2.1 Maxwell distribution

- Maxwell's distribution describes the distribution of free particles on non-bound energy levels (equipartition of kinetic energy)

- For each velocity component, Maxwell's distribution gives the number of particles per unit volume for which the projected velocity on an axis z is between v_z and $v_z + dv_z$

$$n_z dv_z = dn_z = N \left(\frac{m}{2\pi k T} \right)^{1/2} \exp\left(-\frac{m v_z^2}{2 k T} \right) dv_z$$

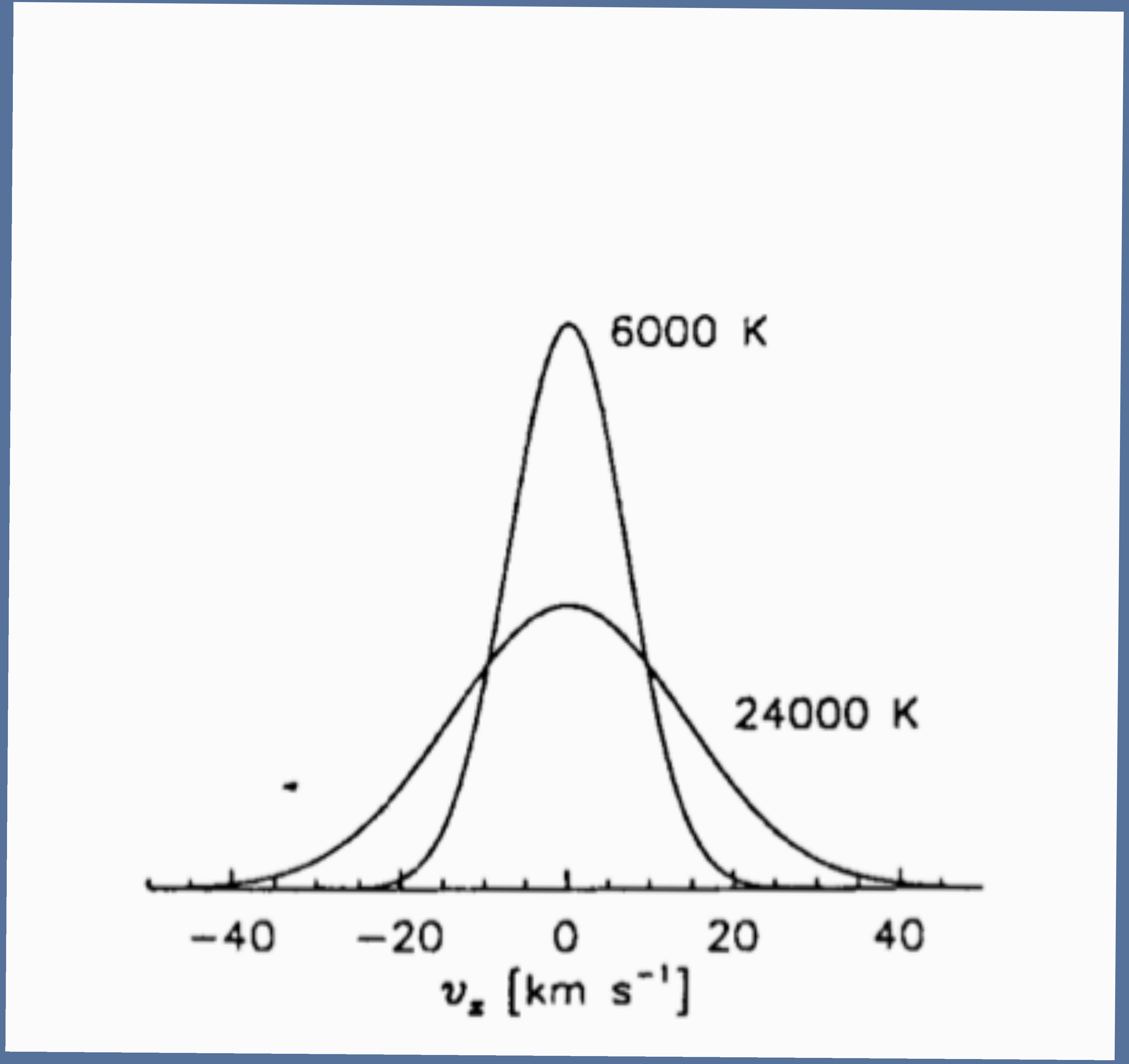
N is the number of particules per unit volume (number density)

m is the particle mass

- This expression is useful to determine the Doppler-broadened line shapes

2.1 Maxwell distribution

- The Maxwell distribution is a Gaussian for which the width depends on the temperature
- The figure on the right shows two such distributions for component v_z, for two different temperatures.



2.1 Maxwell distribution

- The number of particles per unit volume for which the absolute value of the velocity is between $\upsilon = \sqrt{\upsilon_x^2 + \upsilon_y^2 + \upsilon_z^2}$ and $\upsilon + d\upsilon$ (assuming isotropic velocity) is

$$n(\upsilon)d\upsilon = dn(\upsilon) = N\left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi\upsilon^2 \exp\left(-\frac{m\upsilon^2}{2kT}\right)d\upsilon$$

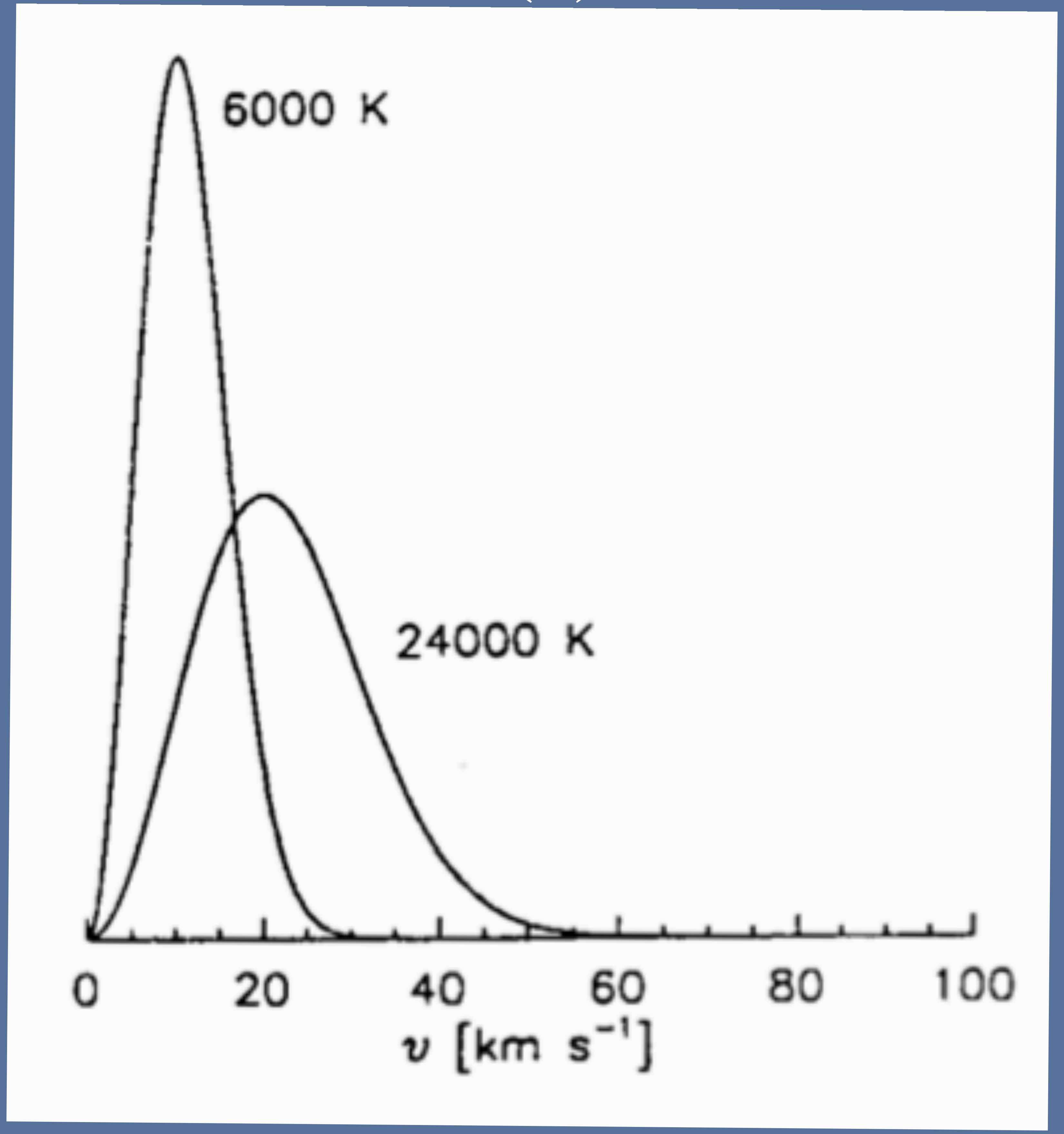
- Because of the term in $4\pi\upsilon^2$, this distribution is no longer a Gaussian (but close to) and has a tail

- Distribution of particules of energy between E and $E + dE$

$$n(E)dE = dn(E) = N\left(\frac{1}{\pi kT}\right)^{3/2} 2\pi\sqrt{E}\exp\left(-\frac{E}{kT}\right)dE$$

- These distributions are normalised: $\int_0^\infty dn = N$

$n(\upsilon)$



2.1 Maxwell distribution

- The most probable velocity is $v = \sqrt{2\,kT/m}$
- What is the most probable value for a velocity component (e.g. v_z)?
- What is the mean energy $< E > = < 1/2\,m\,v^2 >$ of a particle?
- What is the mean Doppler velocity $< v_z^2 >^{1/2}$ along the line of sight?
- For relativistic particles, the distribution has a more complex expression: Maxwell-Jüttner distribution:

$$n(\gamma) = \frac{\gamma^2\,m\,c^2\sqrt{1 - 1/\gamma^2}}{kT\,K_2(mc^2/kT)}\exp\left(-\frac{\gamma\,m\,c^2}{kT}\right)$$

K_2 is the modified Bessel function of the second kind, and $\gamma = 1/\sqrt{1 - v^2/c^2}$

2.2 Boltzmann distribution

- The Boltzmann distribution the distribution of particles of one kind (atoms, molecules, ions) on quantified energy levels. It applies to bound levels.
- For two levels of energy E_u and E_l (measured from ground state) with $E_u > E_l$ and $E_u - E_l = \Delta E_{ul}$, the populations of the levels (number of particles per unit volume) are given by
- $$\frac{n_{u,i}}{n_{l,i}} = \frac{g_{u,i}}{g_{l,i}} \exp\left(-\frac{\Delta E_{ul}}{kT}\right)$$

$$\Delta E_{ul} = h\nu$$

- The index i indicates the ionisation state: the Boltzmann distribution applies to levels in the same ionisation state. In the following, we will omit i.
- This distribution law can apply to electronic, vibrational, or rotational levels
- $g_{u,i}$ and $g_{l,i}$ are the statistical weights of the levels (given in spectroscopic tables)

2.2 Boltzmann distribution

Partition function

- In LTE, we can define the partition function:

$$Z = \sum_j g_j \exp\left(-\frac{E_j}{kT}\right)$$

- The Boltzmann distribution can be expressed as

$$\frac{n_j}{N} = \frac{1}{Z}\, g_j \, \exp\left(-\frac{E_j}{kT}\right)$$

- Note that the partition function is only defined at thermodynamic equilibrium

2.2 Boltzmann distribution

- The energy levels or energy differences between levels are often given in eV (electronic levels or transitions), cm^{-1} (vibrational levels or transitions), or in K (rotational levels or transitions), and not in erg.
- $1\ \mathrm{eV} = 1.602\ 10^{-12}\,\mathrm{erg} = 1.602\ 10^{-19}\,\mathrm{J}$
- Wavenumbers (in cm^{-1}) are defined by $\sigma = c \cdot \nu = c\, E\, /\, h$
- For low energies, the level energy is often given as a temperature such as $T = h\nu/k = E/k$
- Usually, level energies are measured upwards from the ground state within each ionisation state
- The same goes for the ionisation energies
- In some cases though, energies are measured downwards from the ionisation energy
- What energy in eV is associated to a spectral line of wavelength $\lambda = 500\,\mathrm{nm}$?

2.3 Saha equation

- Saha's equation describes the distribution of particles of the same species in different ionisation states.
- For the ground state: $\frac{n_{i+1,0}\, n_e}{n_{i,0}} = 2\, \frac{g_{i+1,0}}{g_{i,0}} \left(\frac{2\pi\, m_e\, kT}{h^2} \right)^{\frac{3}{2}} \exp\left(- \frac{E_i}{kT} \right)$
  - n_e electron density
  - m_e electron mass, $m_e = 9.1\ 10^{-28}\,\mathrm{g}$
  - $n_{i+1,0}$ and $n_{i,0}$, populations (number density of ions) in the ground state of the ionisation degree $i+1$ and i (adjacent ionisation degrees)
  - $g_{i+1,0}$ and $g_{i,0}$ the statistical weights of these levels
  - E_i the ionisation potential of the ion in the i^{th} ionisation degree (required energy to remove one electron of the ion in its ground state and ionisation degree i)
  - The factor 2 comes from the statistical weight of the freed electron (2 possible spin values)

2.3 Saha equation

- Using Boltzmann's law, we obtain, for all levels of the ionisation degree
- $$\frac{n_{i+1}}{n_i} n_e = 2\,\frac{Z_{i+1}}{Z_i}\left(\frac{2\pi\, m_e\, kT}{h^2}\right)^{\frac{3}{2}} \exp\left(-\frac{E_i}{kT}\right)$$
    - n_{i+1} and n_i are the number density of ions in the ionisation degrees $i+1$ and i
    - Z_{i+1} and Z_i are the partition functions of these ionisation states
- At low temperatures, the exponential factor is very small and the partition function is reduced to the statistical weight of the ground state
- At high temperatures, this is not the case anymore. The calculation of the partition function involves a sum over many levels, which is difficult, but essential for stellar atmospheres

2.3 Saha equation

- If $P_e = n_e kT$ is the electron pressure, Saha's equation can be written

  $$\frac{n_{i+1}}{n_i} P_e = 2 \frac{Z_{i+1}}{Z_i} \left(\frac{2\pi m_e}{h^2} \right)^{\frac{3}{2}} (kT)^{\frac{5}{2}} \exp \left(- \frac{E_i}{kT} \right)$$

- With Saha and Boltzmann's laws, it is possible to determine the population ratios for a given element (Saha-Boltzmann distribution)

- In order to determine the number density of particles in a specific state for an arbitrary mix of gas in LTE, one has to use as well
  - Element conservation: $\sum_i n_i = N_{\text{elem}}$
  - Charge conservation: $\sum_{\text{elem}} \sum_i c_i \, n_i = n_e$, with c_i the charge
  - The equations can be solved numerically, by iterations
  - Sometimes only two ionisation levels of one element are interesting (sufficiently populated)
  - One also has to take into account trace element with low ionisation potential. These elements contribute notably to the electronic density n_e

3. Temperatures

- To characterise the energy distribution of particules and photons when LTE is not guaranteed, several temperatures are defined
- At LTE, these temperatures are all equal to the thermodynamic temperature of the medium

3.1 Brightness temperature

- It is the temperature of an equivalent blackbody which would have the same intensity at frequency ν
  - $B_\nu(T_b) = I_\nu^{\text{obs}}$
  - T_b is the brightness temperature
  - T_b obviously depends on the frequency (except at LTE)

3.1 Brightness temperature

- This temperature is widely used in radioastronomy, where the Rayleigh-Jeans approximation applies
  - $T_b = \frac{c^2}{2\nu^2 k} I_\nu^{\mathrm{obs}} = \frac{\lambda^2}{2k} I_\nu^{\mathrm{obs}}$
  - In the Rayleigh Jeans approximation, the brightness temperature is proportional to the observed intensity
  - A temperature is easier to interpret physically
- When the RJ approximation is not quite valid, e.g. in millimeter radioastronomy, especially at low temperatures, we can introduce the Rayleigh-Jeans equivalent temperature $J_\nu(T)$
  - $J_\nu(T) = \frac{c^2}{2\nu^2\, k} I_\nu = \frac{h\nu}{k} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$
  - $J_\nu(T)$ is proportional to the intensity, and has the dimension of a temperature, like in the RJ case

3.1 Brightness temperature

- What would be the definition of the brightness temperature if the intensity is given in unit wavelength? In the general case, and in the Rayleigh-Jeans approximation?
- Does the brightness temperature of a radio source depend on distance?
- Is it possible to measure the brightness temperature of a point source like a star?
- Is it possible to measure the brightness temperature of an extended source (like a nebula) if it is not at thermodynamic equilibrium?

3.2 Excitation temperature

- The excitation temperature T_{ex} is defined as $S_\nu = B_\nu(T_{\mathrm{ex}})$
- It is the temperature of an equivalent blackbody with the intensity equal to the source function at frequency ν
- Application for a homogeneous medium
  - Solution of the RT equation: $I_\nu = I_\nu(0)\, \mathrm{e}^{-\frac{\tau_\nu}{\mu}} + S_\nu\, (1 - \mathrm{e}^{-\frac{\tau_\nu}{\mu}})$
  - In the RJ case: $T_b = T_b(0)\, \mathrm{e}^{-\frac{\tau_\nu}{\mu}} + \dfrac{\lambda^2}{2k} S_\nu\, (1 - \mathrm{e}^{-\frac{\tau_\nu}{\mu}})$
  - With the definition of T_{ex}: $T_b = T_b(0)\, \mathrm{e}^{-\frac{\tau_\nu}{\mu}} + T_{\mathrm{ex}}\, (1 - \mathrm{e}^{-\frac{\tau_\nu}{\mu}})$
  - With the radiation temperature:

$$J_\nu(T_b) = J_\nu(T_b(0))\, \mathrm{e}^{-\frac{\tau_\nu}{\mu}} + \frac{\lambda^2}{2k} S_\nu\, (1 - \mathrm{e}^{-\frac{\tau_\nu}{\mu}}) = J_\nu(T_b(0))\, \mathrm{e}^{-\frac{\tau_\nu}{\mu}} + J_\nu(T_{\mathrm{ex}})\, (1 - \mathrm{e}^{-\frac{\tau_\nu}{\mu}})$$

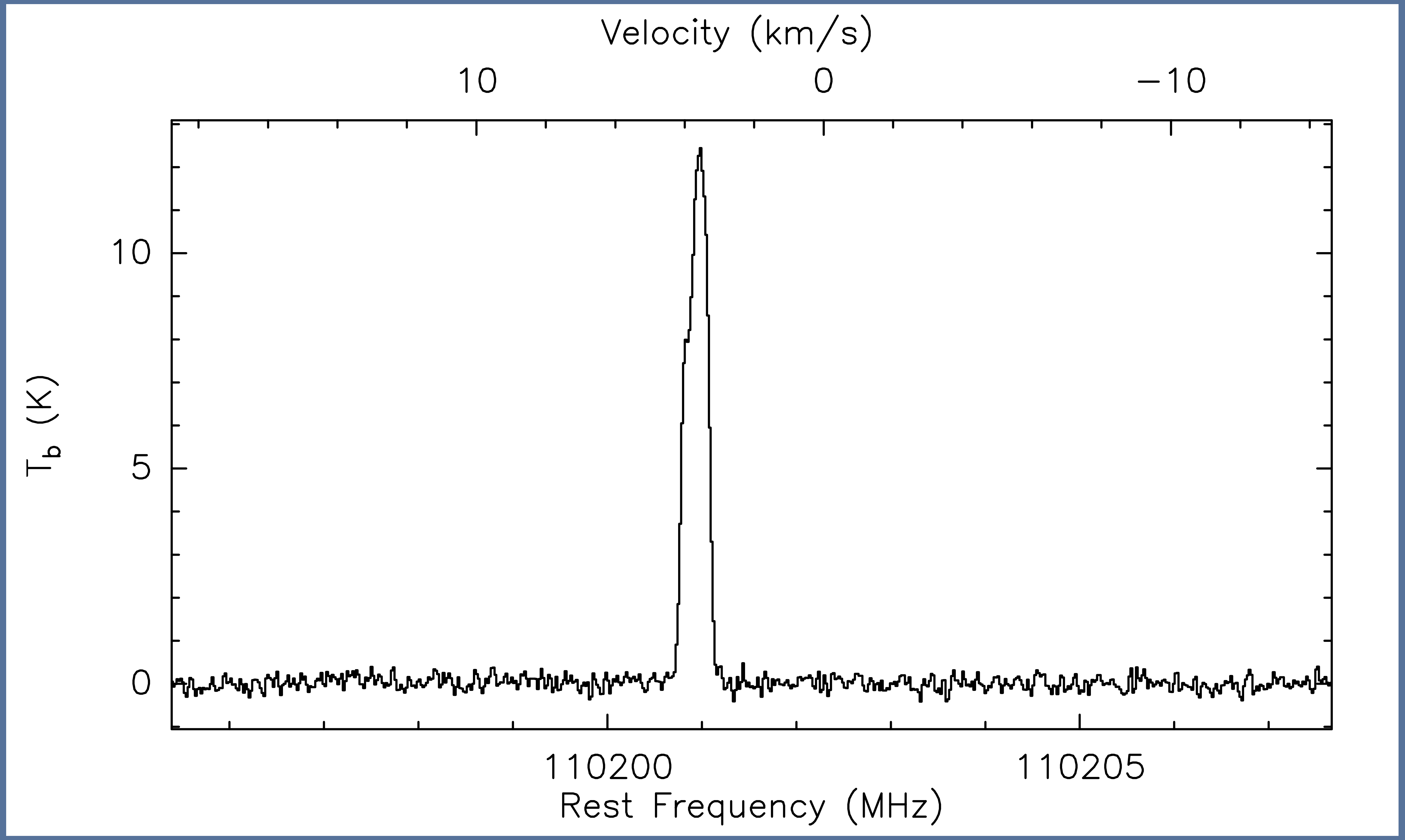
3.2 Excitation temperature

- $T_b = T_b(0)\,\mathrm{e}^{-\frac{\tau_\nu}{\mu}} + T_{\mathrm{ex}}\,(1 - \mathrm{e}^{-\frac{\tau_\nu}{\mu}})$
- Optically thick case: $T_b = T_{\mathrm{ex}}$
- Optically thin case: $T_b = T_b(0) + \tau\, T_{\mathrm{ex}}$
- LTE case: $S_\nu = B_\nu(T) = B_\nu(T_{\mathrm{ex}}) \Rightarrow T_{\mathrm{ex}} = T$
  - the excitation temperature is equal to the thermodynamic temperature of the medium
- LTE and optically thick case: $T_b = T$
  - the brightness temperature is equal to the thermodynamic temperature of the medium

3.2 Excitation temperature

- These considerations show that for LTE and in the optically thick case, we measure directly the kinetic temperature of the medium

^{13}CO(1-0) spectrum in a dark cloud



^{13}CO at LTE, optically thick
➜ gas is at T_k ~ 13 K (RJ limit)

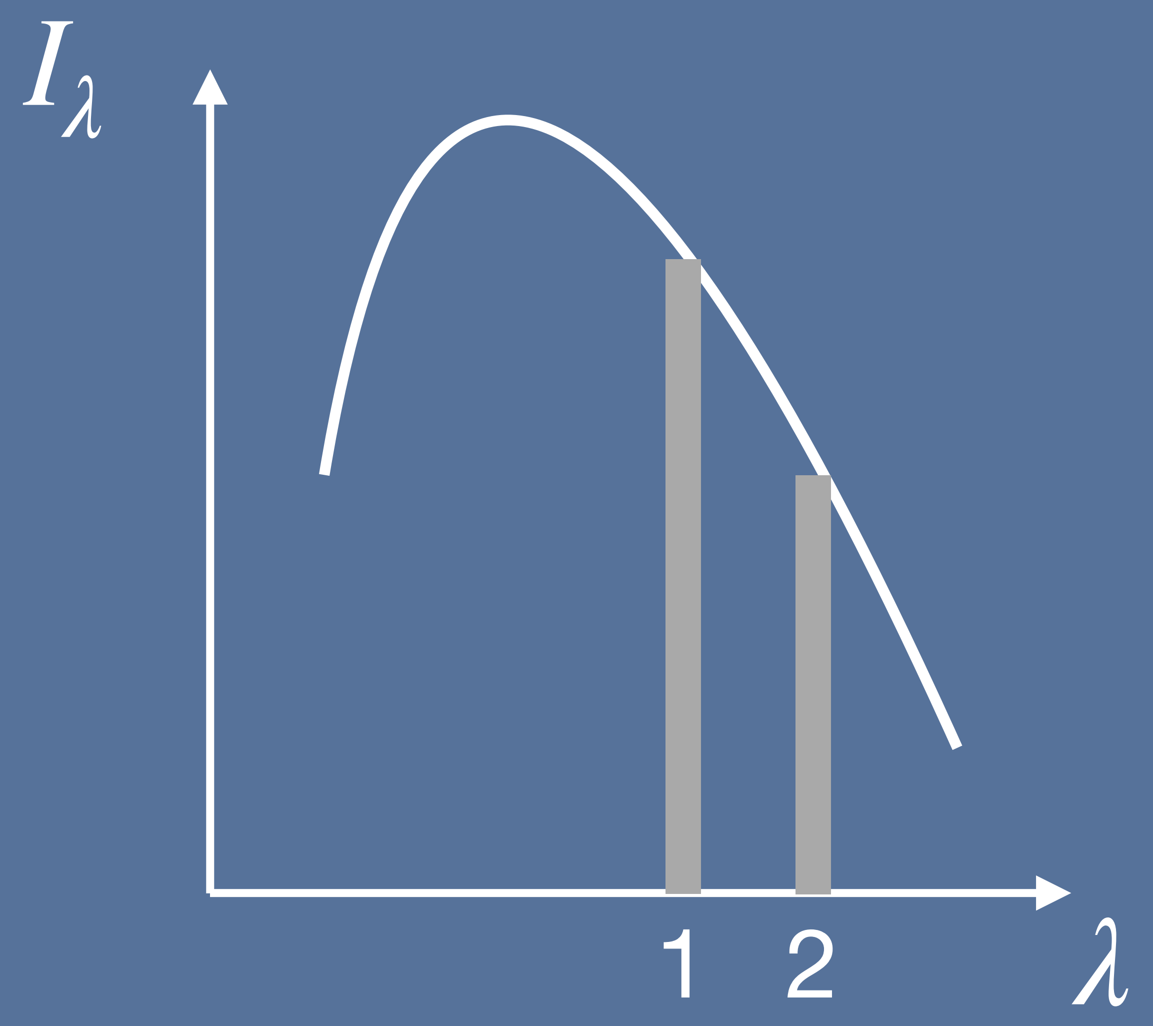
in fact RJ does not apply, so $J_\nu(T_k)$ = 13 K
➜ gas is at T_k ~ 15 K

3.3 Colour temperature

- It is the temperature of the equivalent blackbody which has the same intensity variation with frequency as the intensity emerging from the medium
- It is the temperature for which Planck's law reproduces the slope of the intensity spectrum at the observing frequency
- The colour temperature T_c verifies $\left.\frac{dI_\nu}{d\nu}\right|_{\nu=\nu_0} = \left.\frac{B_\nu(T_c)}{d\nu}\right|_{\nu=\nu_0}$

3.3 Colour temperature

- Another definition is often used, in particular in the case of 2-colour photometry: $\dfrac{I_{\lambda_1}}{I_{\lambda_2}} = \dfrac{B_{\lambda_1}(T_c)}{B_{\lambda_2}(T_c)}$
- The ratio between the 2 observed intensities defines a temperature
- This is also expressed in magnitudes: $B - V = 2.5 \log \dfrac{I_V}{I_B}$
- Can we apply two-band photometry to stars, knowing that stars are unresolved sources?
- What conditions are necessary for two-band photometry of a star to give its temperature? Of which stellar region is it the temperature?



3.3 Colour temperature

- In an unresolved sources, we can only measure the flux and not the intensity
- For a star of radius R at a distance d, of source function S, the ratio of the observed fluxes is

$$\frac{F_1^{\text{obs}}}{F_2^{\text{obs}}} = \frac{F_1^+ R_1^2/d^2}{F_2^+ R_2^2/d^2} = \frac{\pi S_1(\tau_1 = 2/3)}{\pi S_2(\tau_2 = 2/3)} = \frac{I_1(\tau_1 = 0, \mu = 2/3)}{I_2(\tau_2 = 0, \mu = 2/3)}$$

- With $R_1 = R_2$ and if the Eddington-Barbier approximation is valid at both frequencies.
- The flux ratio therefore gives the intensity ratio of the star at both frequencies
- The advantage of using the flux ratio $\frac{F_1^{\text{obs}}}{F_2^{\text{obs}}} = \frac{B_{\nu_1}(T_c)}{B_{\nu_2}(T_c)}$ is that it is not necessary to know the distance to the source. In such a case, T_c is a good replacement of the brightness temperature

3.4 Effective temperature

- It is the temperature of the equivalent blackbody which radiates the same flux integrated over the frequency
- $F = \int F_\nu d\nu = \sigma T_{\text{eff}} = \pi\, B_\nu(T_{\text{eff}})$
- T_{eff} is the effective temperature
- It can be expressed as function of the intensity emerging from a spherically symmetric source
- $T_{\text{eff}} = [\frac{1}{\sigma}\, F]^{1/4} = [\frac{2\pi}{\sigma} \int_0^\infty \int_0^1 \mu\, I_\lambda\, d\mu\, d\lambda]^{1/4}$
- At thermal equilibrium, $T_{\text{eff}} = T$