Radiative Transfer

5. Introduction to scattering

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Introduction

- We saw in Chapter 3 the general form of the transfer equation: $\frac{dI_\nu(\vec{n}, s)}{ds} = j_\nu(s) - \alpha_\nu I_\nu(\vec{n}, s)$
- If we knew at each location and each time j_ν and α_ν, we would already have everything at hand to understand radiative transfer
- We would of course still need to discuss physical parameters like opacities and abundances of the medium constituents, but there is no overwhelming difficulty
- The reason why radiative transfer is a difficult subject is that in most cases, we do not know the values of j_ν and α_ν in advance
- The radiative field that we would like to determine can affect the medium and modify j_ν and α_ν. We face the problem of the chicken and the egg: to calculate $I_\nu(\vec{x}, \vec{n})$, we have to know $j_\nu(\vec{x})$ and $\alpha_\nu(\vec{x})$ and to know $j_\nu(\vec{x})$ and $\alpha_\nu(\vec{x})$, we have to know $I_\nu(\vec{x}, \vec{n})$.

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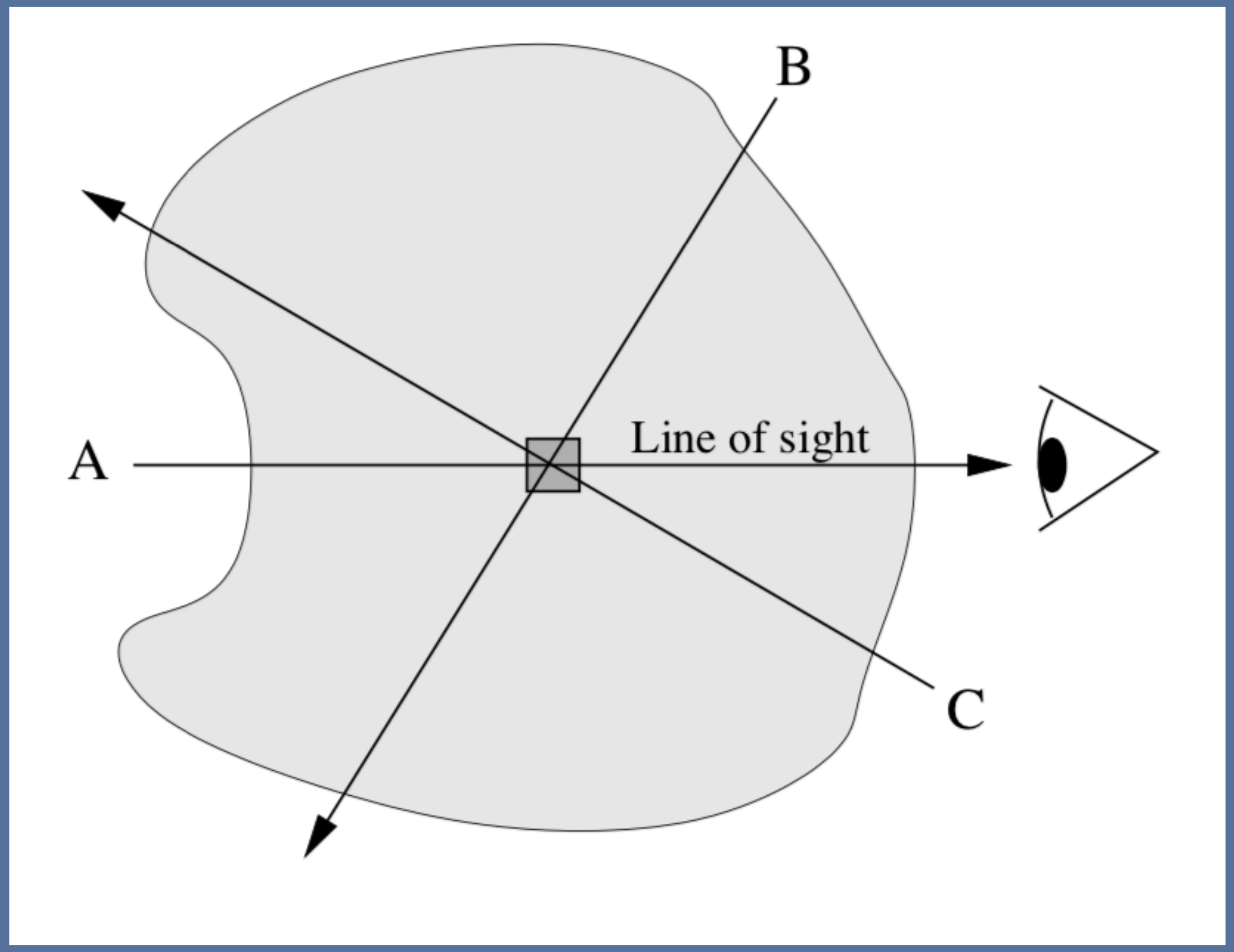
- Even worse, we cannot solve the problem separately for each ray because a change in $j_\nu(\vec{x})$ affects the transfer equation for all rays passing through $\vec{x}$, even if their directions $\vec{n}$ are different.

- If for example we are interested in ray A (along the line of sight): at location $\vec{x}$ represented by a small square, we have $j_\nu(\vec{x})$ and $\alpha_\nu(\vec{x})$. Ray A goes through this volume element, but also rays B and C. The intensities along rays B and C therefore also affect $j_\nu(\vec{x})$ and $\alpha_\nu(\vec{x})$



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- This coupling between rays means that we must solve the radiative transfer problem for all rays simultaneously. This is a challenge of radiative transfer

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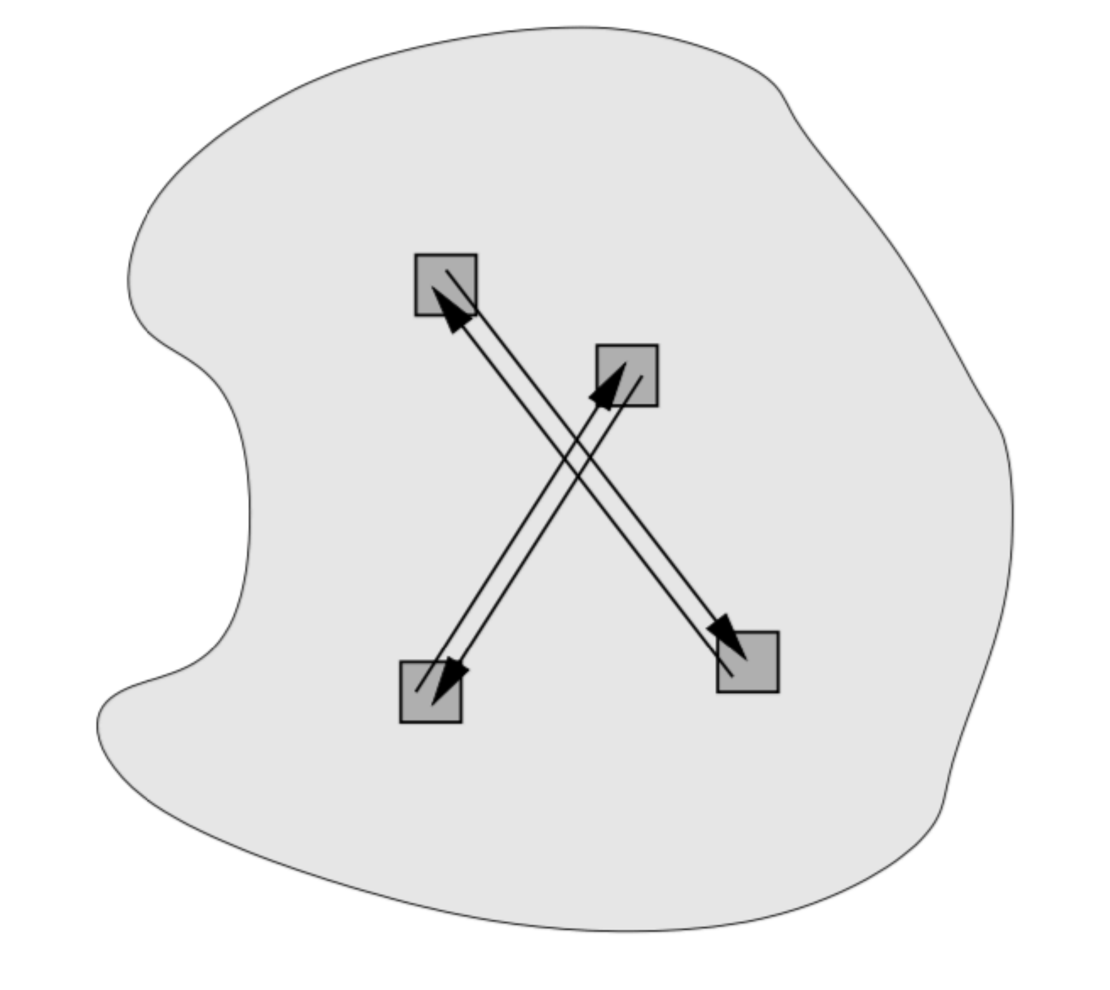
- We can also consider that all volume elements that make up the medium are radiatively coupled (radiative cell coupling)

- The emission of one volume element can affect the conditions in another volume element in the medium

- There is an exchange in information between regions in the medium that are distant from one another. For example, the radiative cooling of one region can lead to the radiative heating of another



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Introduction

- In this chapter, we will deal with scattering, which is another difficulty in radiative transfer (especially multiple scattering).
- Scattering is a good example of the non-local character of radiative transfer
- At LTE, the source function is determined locally, because of the coupling between particle and radiative energy. If collisional processes do not dominate but scattering does instead, the local character is lost: the photons that are scattered come from elsewhere
- The mean free path of a photon between two extinctions is $l_{\rm free} = 1/\alpha_\nu$. If extinctions processes are mostly elastic scattering events, the identity of the scattered photon remains the same. The photon changes direction each time it is scattered, but not energy.
- The distance between photon creation and destruction or between photon creation and when it escapes the medium can be much larger than $l_{\rm free}$

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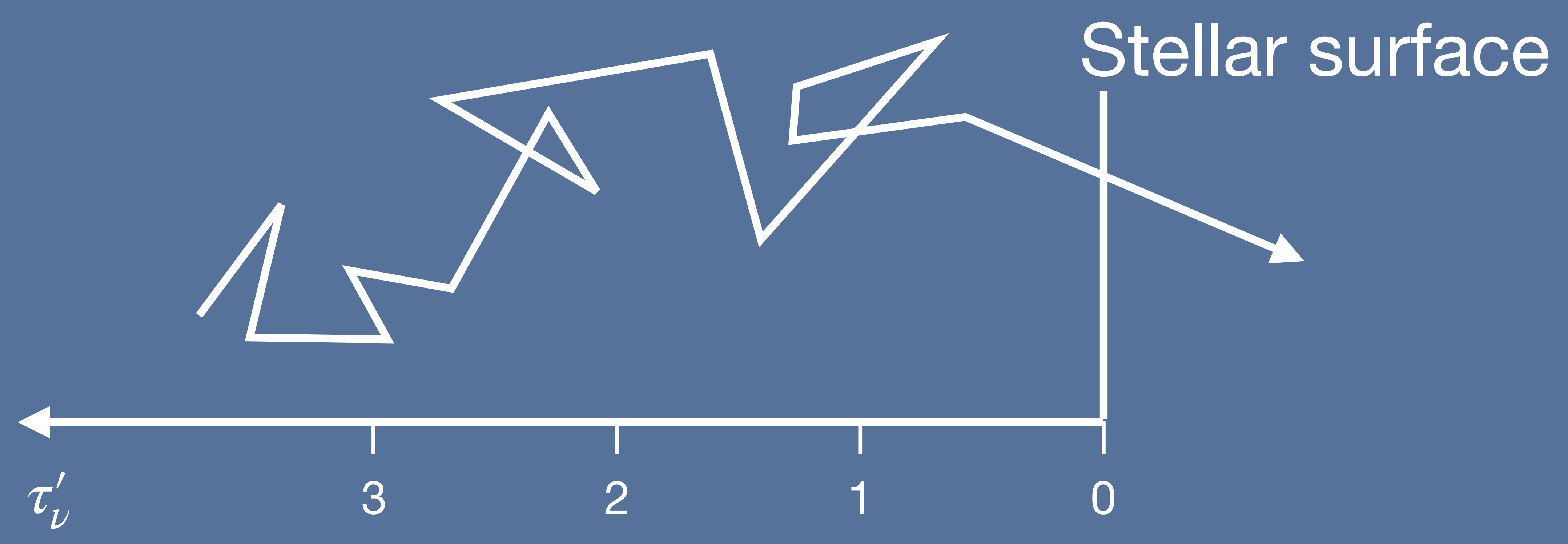
- For example in a stellar atmosphere



- The photons emerge from an optical depth $\tau'_\nu = 1$ (or $\tau'_\nu = \mu$ if τ'_ν is the radial optical depth), but this optical depth is that of the photons' last interaction, where they were scattered.
- However the depth at which they were created can be much greater. From that point they move according to a random walk
- The exact scattering process does not matter: Thompson, Rayleigh, Mie, elastic bound-bound

Introduction

- According to the Eddington-Barbier relation, the photons that escape correspond to a source function at an optical depth of $\tau_\nu' = \mu$. Is this also valid in the case of scattering?

- The Eddington-Barbier approximation applies to the total optical depth, independently of the nature of the extinction (absorption or scattering). What we see is where the radiation escapes and not where it was created.

- See the example of Chapter one with the lamp in the fog.

1. Pure isotropic scattering

- It is the simplest radiative transfer problem with ray coupling.
- We assume a medium made of particules (for example, small dust grains) that can scatter the radiation in arbitrary directions.
- This process is called "isotropic scattering" because the direction of the emerging photon does not depend on the direction of the incident photon before the scattering event.
- We will also assume that the particles do not absorb and do not emit radiation, and that we have coherent scattering, ie without frequency change.
- We consider the monochromatic case (only one frequency ν, which we do not write)
- Inside or outside of the medium, there is a light source

1.1 Radiative transfer equation for scattering

- The formal equation is identical to the one given previously
- $\overrightarrow{n} \cdot \overrightarrow{\nabla} I(\overrightarrow{x}, \overrightarrow{n}) = j(\overrightarrow{x}) - \alpha(\overrightarrow{x})\, I(\overrightarrow{x}, \overrightarrow{n})$
  - j is the emissivity, ie the photon injection by scattering into the direction of propagation $\overrightarrow{n}$ of the radiation
  - α is the extinction linked to the scattering only
  - The extinction coefficient for scattering is sometimes written σ_ν, with the same dimension as α_ν (cm^{-1}), but it can also be defined per mass unit. Here we will use α_ν or $\alpha_\nu^{\mathrm{sca}}$ as is usual in stellar physics. The total extinction coefficient is $\kappa_\nu + \sigma_\nu$, or $\alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{sca}}$
- All scattered photons have the same probability to be scattered in the direction $\overrightarrow{n}$, so that we only need to know which amount of radiation is scattered per unit volume and unit time

1.1 Radiative transfer equation for scattering

- $\alpha_\nu I_\nu \,(= \alpha_\nu^{\rm sca} I_\nu)$ is the power scattered by unit solid angle, by unit frequency during the crossing of a layer of unit length perpendicular to the direction of propagation

- The power reemitted by scattering per unit solid angle per unit frequency in the direction of propagation is: $j_\nu(\vec{x}) = \alpha_\nu(\vec{x}) \dfrac{1}{4\pi} \oint I_\nu(\vec{x}, \vec{n})\, d\Omega = \alpha_\nu(\vec{x})\, J_\nu(\vec{x})$

- The last relation is just the definition of $J_\nu(\vec{x})$. The transfer equation is therefore:

- $\vec{n} \cdot \vec{\nabla} I_\nu(\vec{x}, \vec{n}) = \alpha(\vec{x}) \left[\dfrac{1}{4\pi} \oint I_\nu(\vec{x}, \vec{n}')\, d\Omega' - I_\nu(\vec{x}, \vec{n}) \right]$

- Or in a more compact way: $\vec{n} \cdot \vec{\nabla} I_\nu(\vec{x}, \vec{n}) = \alpha(\vec{x})\, [J_\nu(\vec{x}) - I_\nu(\vec{x}, \vec{n})]$

1.1 Radiative transfer equation for scattering

- The equation has exactly the same form as previously, with the source function $S_\nu = J_\nu$.
- The main difference is that we now have an integro-differential equation, which is very difficult to solve
- The equation illustrates very well the problem of the chicken and the egg which makes radiative transfer so hard to solve: we need to know $J_\nu(\overrightarrow{x})$ in order to integrate the equation in one direction and obtain $I_\nu(\overrightarrow{x}, \overrightarrow{n})$, but we need to know $I_\nu(\overrightarrow{x}, \overrightarrow{n})$ for all directions to calculate $J_\nu(\overrightarrow{x})$.

1.2 Multiple scattering

- Let's take the inverse path of the light back to the source.
- The photons we observe have been scattered in the direction of the line of sight by one particle.
- Before that, they were moving along another ray, but they may have been scattered into this ray by another scattering event on another particle, and so on.
- Photons will be scattered several times before being scattered in the line of sight.
- In order to understand the problem posed by multiple scattering, we can think recursively: each scattering event is in fact a "chicken-egg" cycle

1.2 Multiple scattering

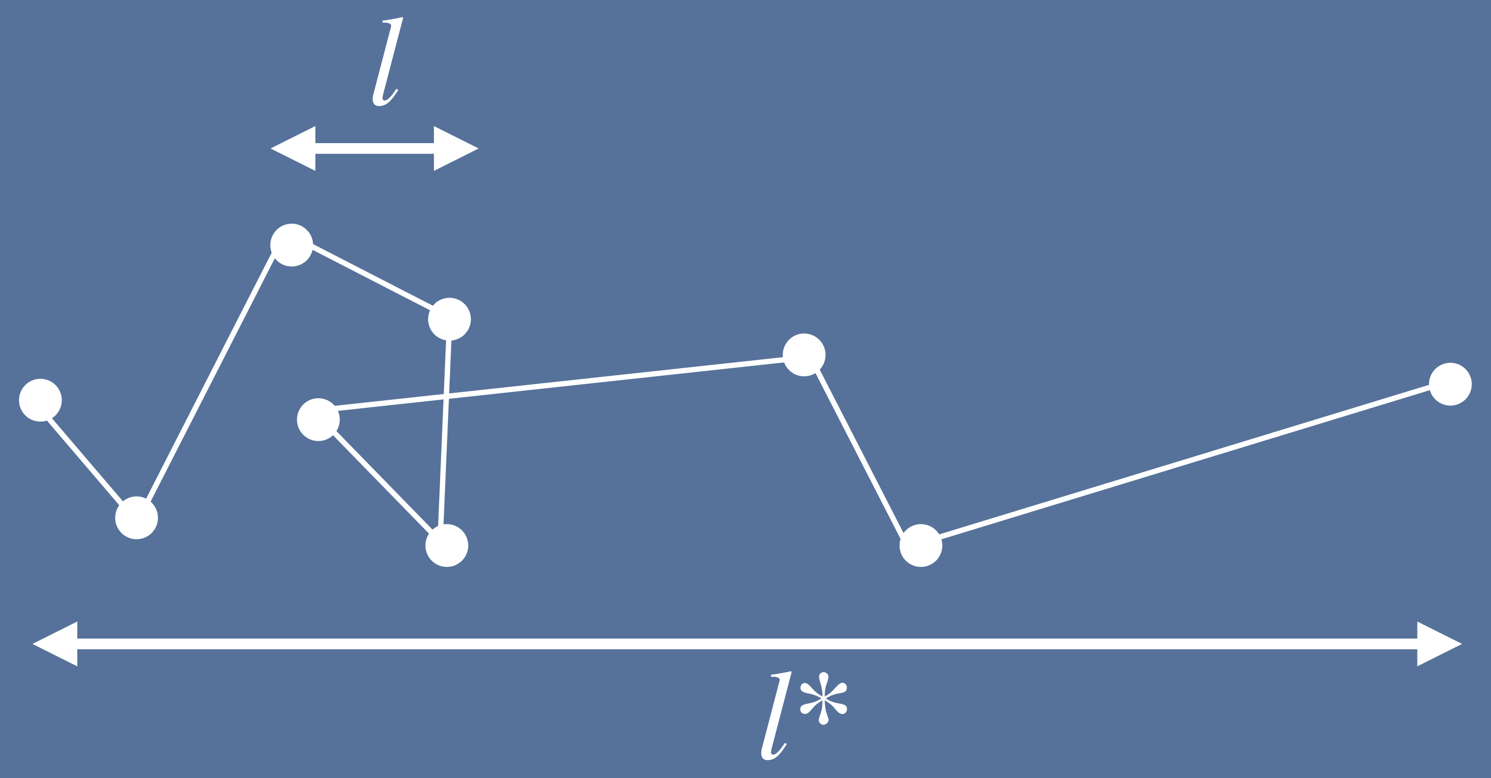
- To calculate J at a location $\overrightarrow{x_0}$ along the line of sight, we have to integrate the transfer equation for all rays that pass through $\overrightarrow{x_0}$, ie by varying $\overrightarrow{n}$ over 4π sr.
- But to integrate the transfer equation along all these rays, we need to know J at all other locations $\overrightarrow{x} \neq \overrightarrow{x_0}$ along these rays, which implies integrating the radiative transfer equation for all these rays that pass through $\overrightarrow{x}$, by varying $\overrightarrow{n}$ over 4π sr, and so on.
- How can we solve this?
- Analytical solutions are very rare. One of them was given by Chandrasekhar ("Radiative Transfer", 1950/1960, Dover) for a semi-infinite homogeneous plane parallel atmosphere.
- In most cases however, numerical methods are used.

1.2 Multiple scattering

- What is the distance $l*$ crossed by a photon after N scattering events?

$$l* \simeq \sqrt{N}\, l$$

For a derivation see Rybicki & Lightman



- How many scattering events does it take for the photon to cross a medium of thickness D?

- $l*$ has to be equal to D, ie $N = \left(\frac{l*}{l}\right)^2 \sim \frac{D^2}{l^2}$

- with $l = \frac{1}{\alpha_\nu} = \frac{1}{\alpha_\nu^{\rm sca}}$ for pure scattering and $\tau_\nu = \alpha_\nu D$, we obtain $N \simeq \tau_\nu^2$ if $\tau_\nu \gg 1$ (for the photon to be scattered)

- For an optically thin medium with $\tau_\nu \ll 1$, the photon generally escapes immediately, with a small interaction probability roughly equal to $\tau_\nu = \alpha_\nu D \ll 1$

1.3 Approximation of single scattering

- For $\tau_\nu \ll 1$, it is possible to make an approximation so that the problem can be solved analytically, or with minimum numerical effort
- In this case, we can ignore multiple scattering and assume that each photon that is scattered towards the line of sight had not been scattered before
- This is the approximation of single scattering. The lower the optical depth of the medium, the better the approximation.

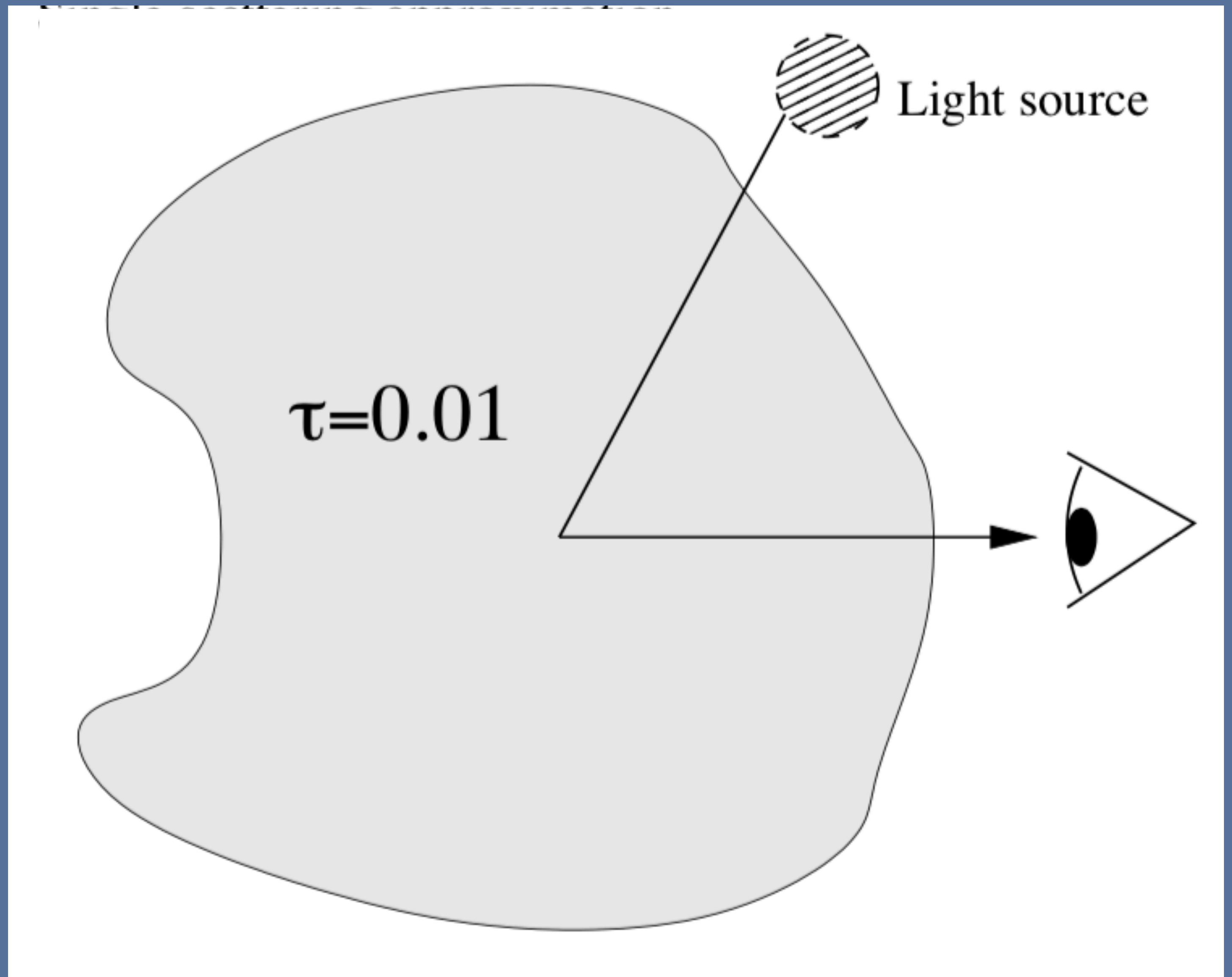
1.3 Approximation of single scattering

- For single scattering, we have to

  1. Integrate the radiative transfer equation for all rays joining the light source to the line of sight (ie determine I_ν along all those directions)

  2. Calculate j_ν at all locations along the line of sight

  3. Integrate the formal transfer equation along the observer's line of sight



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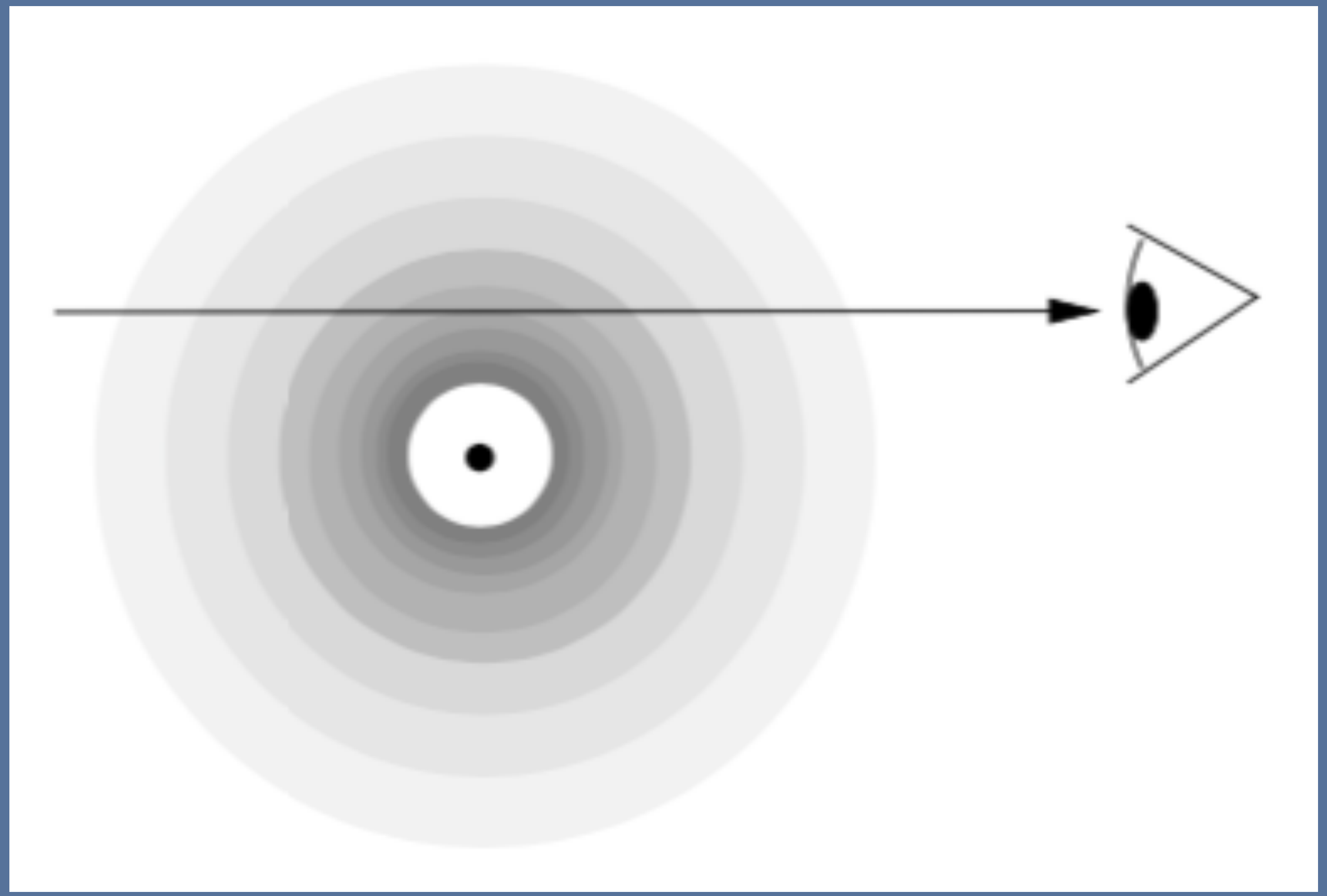
- Even though the integration of the transfer equation along all the rays connecting the source and the line of sight can be difficult or necessitates some computing time, it remains generally doable.

1.3 Approximation of single scattering

- Example of single scattering: a star surrounded by a spherical cloud
    - star: radius R_*, temperature T_*, radiates like a perfect blackbody
    - A spherical dust cloud surrounds the star. The cloud density is given by

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^{-2} \quad \text{pour} \quad r \geq r_0$$

$$\rho(r) = 0 \quad \text{pour} \quad r < r_0$$



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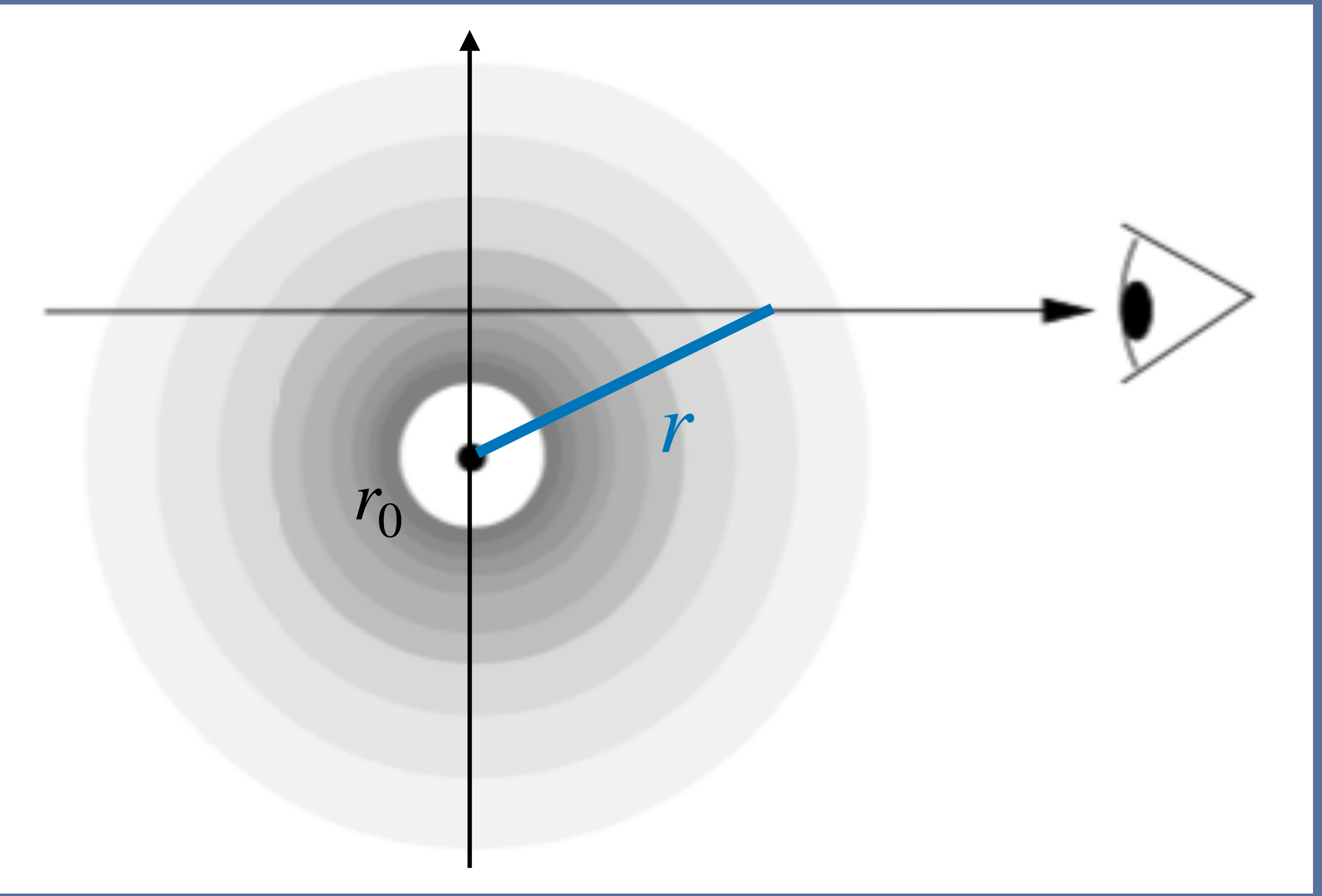
- We also assume that the scattering opacity does not depend on the frequency, density or temperature: $\kappa_\nu = \kappa$ (mass extinction coefficient)

1.3 Approximation of single scattering

- We assume that the optical depth between the star and a point at a distance r is small enough to use the single scattering approximation: $\tau_\nu(r) = \kappa_\nu \int_{r_0}^{r} \rho(r')\, dr' \ll 1$
- $r_0 \gg R_*$ and the star can be considered as a point source
- The flux of the star is $F_\nu(r) = \dfrac{L_\nu}{4\pi\, r^2}$, with $L_\nu = 4\pi\, R_*^2\, \pi\, B_\nu(T_*)$
- To calculate the emissivity j_ν along the line of sight, we need J_ν which is $J_\nu = \dfrac{F_\nu(r)}{4\pi}$ for a ray pointing exactly outward
- The emissivity is therefore: $j_\nu(r) = \alpha_\nu \dfrac{F_\nu}{4\pi} = \dfrac{1}{(4\pi)^2} \kappa_\nu\, L_\nu\, \rho_0\, r_0^2\, \dfrac{1}{r^4}$



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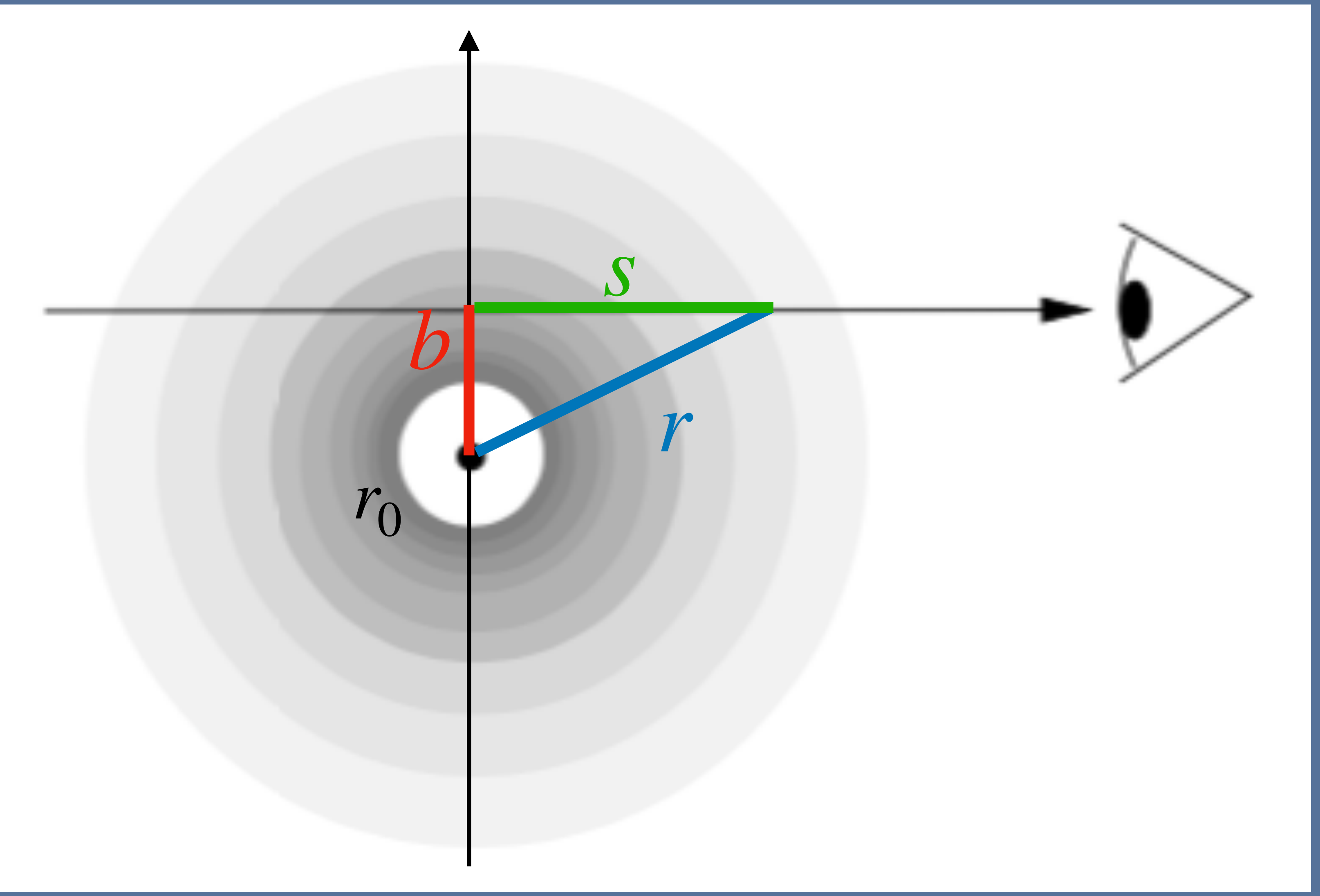
1.3 Approximation of single scattering

- We now need to integrate the emissivity along the line of sight: we note b the impact parameter and s the coordinate along the line of sight with $s = 0$ the closest point to the star.



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- $r = \sqrt{b^2 + s^2}$

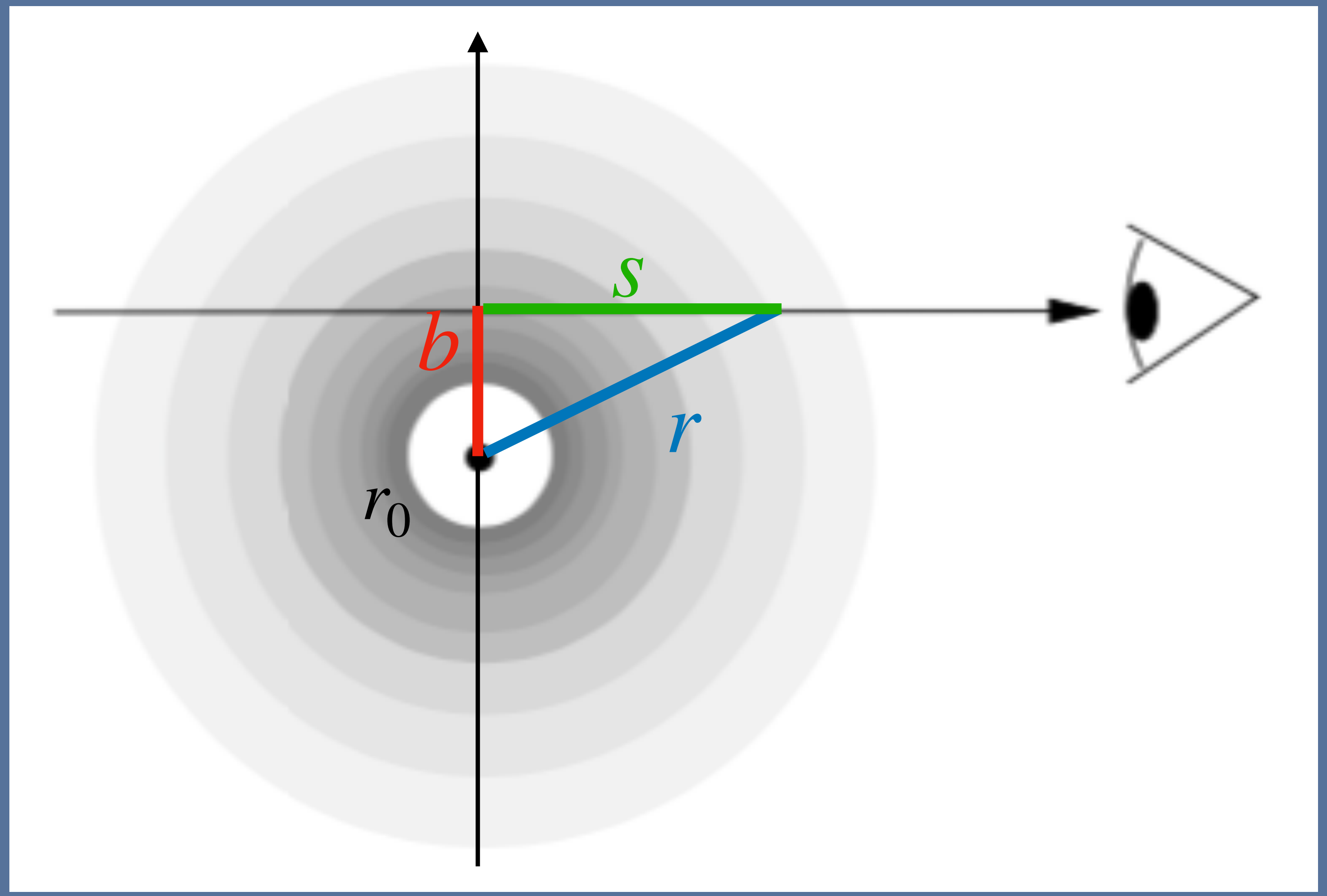
- The integral along the line of sight is

$$I_\nu^{\rm obs}(b) = \frac{1}{(4\pi)^2}\,\kappa_\nu\,L_\nu\,\rho_0\,r_0^2 \int_{-\infty}^{+\infty} \frac{ds}{(b^2+s^2)^2} = \frac{1}{(4\pi)^2}\,\kappa_\nu\,L_\nu\,\rho_0\,\frac{r_0^2}{b^3} \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2}$$

having defined $x = \dfrac{s}{b}$

1.3 Approximation of single scattering

- $$I_\nu^{\mathrm{obs}}(b) = \frac{1}{32\pi}\,\kappa_\nu\,L_\nu\,\rho_0\,\frac{r_0^2}{b^3}$$
- in the case without background intensity
- The intensity is $\int_{-\infty}^{+\infty} j_\nu(s)\,ds$ without extinction because we have assumed that photons are only scattered once and they have already been scattered into the line of sight



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- On an image, we would see the intensity of the scattered radiation decrease in $1/b^3$ away from the star. The profile in $\rho \propto 1/r^2$ for the density is what is expected for a stellar wind. In reality, isotropic scattering is not always a good approximation for scattering on dust particles (see Chapter 6) but the decrease in $1/b^3$ even for anisotropic scattering is not a bad approximation

2. Scattering with absorption and thermal emission

- The previous problem, isotropic scattering, is extremely hard to solve but has been somewhat idealised: the hypothesis we made is that the particles do not absorb any radiation (they only scatter it) and do not emit any (no thermal radiation from dust for example)
- for water droplets in the Earth atmosphere in the visible, this is a reasonable approximation
- But often, in addition to scattering, there is also thermal emission and absorption. It is the case for atmospheric aerosols for example
- In astrophysical problems, there are many cases in which at the same time absorption, emission, and scattering play a role.

2.1 Albedo and photon destruction probability

- If we consider absorption and scattering, we have for extinction two contributions at frequency ν: $\alpha_\nu = \alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}$
- We can define the albedo: $\eta_\nu = \dfrac{\alpha_\nu^{\rm sca}}{\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}}$
- Watch out for notations: some authors denote the albedo with α, which we will not use so as to avoid confusion with the extinction coefficient
- We can also define the photon destruction probability: $\epsilon_\nu = \dfrac{\alpha_\nu^{\rm abs}}{\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}}$
- η_ν is the scattering probability per extinction, and ϵ_ν the photon absorption probability per extinction
- These quantities are widely used in non-LTE RT theory for stellar atmospheres

2.1 Albedo and photon destruction probability

- Obviously we have $\epsilon_\nu = 1 - \eta_\nu$
- The emissivity is also the sum of two contributions: $j_\nu = j_\nu^{\rm emi} + j_\nu^{\rm sca}$
- $j_\nu^{\rm emi}$ corresponds to photon creation
- The source function is $S_\nu = \dfrac{j_\nu}{\alpha_\nu} = \dfrac{j_\nu^{\rm emi} + j_\nu^{\rm sca}}{\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}} = \epsilon_\nu \dfrac{j_\nu^{\rm emi}}{\alpha_\nu^{\rm abs}} + \eta_\nu \dfrac{j_\nu^{\rm sca}}{\alpha_\nu^{\rm sca}}$
- This can be rewritten: $S_\nu = \epsilon_\nu\, S_\nu^{\rm abs} + \eta_\nu\, S_\nu^{\rm sca}$
- For isotropic scattering, we had $S_\nu^{\rm sca} = \dfrac{j_\nu^{\rm sca}}{\alpha_\nu^{\rm sca}} = J_\nu$
- For thermal emission at temperature T, we have $S_\nu^{\rm abs} = \dfrac{j_\nu^{\rm emi}}{\alpha_\nu^{\rm abs}} = B_\nu(T)$

2.1 Albedo and photon destruction probability

- The source function is then: $S_\nu = \epsilon_\nu \, B_\nu(T) + \eta_\nu \, J_\nu = \epsilon_\nu \, B_\nu(T) + (1 - \epsilon_\nu) \, J_\nu$ which is the standard notation in the stellar atmosphere community
- The transfer equation remains unchanged: $\dfrac{dI_\nu}{ds} = \alpha_\nu \, [S_\nu - I_\nu]$
- Which can also be written, inserting what precedes: $\dfrac{dI_\nu}{ds} = \alpha_\nu \, [\epsilon_\nu \, B_\nu(T) + (1 - \epsilon_\nu) \, J_\nu - I_\nu]$
- If $\epsilon = 1$, assuming we know the temperature everywhere, there is no longer a chicken-egg problem. The source function is the Planck function.
- The problem is moderately complex for $0 < \epsilon < 1$. In the case $\epsilon = 0.5$, the photon can only be scattered a few times before being destroyed by absorption. The radiative information is on average carried across a few mean free paths before disappearing
- The problem is greatest when 0: the source function is then equal to the intensity averaged over the directions, ie J_ν. The photon is scattered until it escapes the medium, crossing macroscopic distances in the medium
- Generally, the closer ϵ_ν is to 0, the more difficult it is to solve the transfer problem.

2.1 Albedo and photon destruction probability

- In stellar physics, J_ν is the reservoir term, ie the amount of available photons.
- The term $\epsilon_\nu J_\nu$ is the photon sink. It specifies the energy/amount of photons that disappear from the reservoir per extinction.
- The term $\epsilon_\nu B_\nu(T)$ is the source term: it specifies the energy/amount of photons newly created per extinction. This source term cannot be neglected because otherwise no photon would be created (and be scattered).
- As a consequence, even if ϵ_ν is very small, $\epsilon_\nu B_\nu(T)$ must always be precisely evaluated: this term contributes to the radiative term J_ν with which the source function is determined for the most part.

2.2 Effective optical depth

- The mean free path between two successive extinction events for a photon that follows a random walk is

  $$l_\nu = \frac{1}{\alpha_\nu} = \frac{1}{\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}}$$

  but in the presence of scattering, it is more interesting to know over which distance the identity of the photon is conserved, ie what is the path length between its creation and its destruction.

- The absorption probability per step is ϵ_ν, ie the number of steps (of scattering events) that the photon can have is: $N = \dfrac{1}{\epsilon_\nu}$

- From the previously determined relation $l_\nu^* \simeq \sqrt{N}\, l_\nu$, we obtain: $l_\nu^* \simeq \dfrac{l_\nu}{\sqrt{\epsilon_\nu}}$

2.2 Effective optical depth

- l_ν^* is the characteristic distance between photon creation and destruction, ie the distance of conservation of photon identity
- It is also called the diffusion length, or the thermalisation length, or the effective mean free path
  - For $\epsilon_\nu = 1$ ($\alpha_\nu^{\rm sca} = 0$, no scattering), we have $l_\nu^* = l_\nu$
  - For $\epsilon_\nu \ll 1$ ($\alpha_\nu^{\rm sca} \gg \alpha_\nu^{\rm abs}$, a lot of scattering), we have $l_\nu^* \gg l_\nu$
  - For $\epsilon_\nu = 0$ ($\alpha_\nu^{\rm abs} = 0$, pure scattering), we have $l_\nu^* = \infty$
- From $l_\nu = \dfrac{1}{\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}}$ and $\epsilon_\nu = \dfrac{\alpha_\nu^{\rm abs}}{\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}} \quad \Rightarrow \quad l_\nu^* \simeq \dfrac{1}{\sqrt{\alpha_\nu^{\rm abs}\left(\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}\right)}}$

2.2 Effective optical depth

- Four optical depths can be defined
  - The total optical depth τ_ν such as $d\tau_\nu = (\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}})\, ds$
  - The optical depth due to absorption τ_ν^{abs} such as $d\tau_\nu^{\text{abs}} = \alpha_\nu^{\text{abs}}\, ds$
  - The optical depth due to scattering τ_ν^{sca} such as $d\tau_\nu^{\text{sca}} = \alpha_\nu^{\text{sca}}\, ds$
  - The effective optical depth τ_ν^{eff} such as $d\tau_\nu^{\text{eff}} = \sqrt{\alpha_\nu^{\text{abs}}\left(\alpha_\nu^{\text{abs}} + \alpha_\nu^{\text{sca}}\right)}\, ds$
- For a homogeneous medium of thickness D, the effective optical depth τ_ν^{eff} is

$$\tau_\nu^{\text{eff}} = \frac{D}{l_\nu^*} \simeq \sqrt{\tau_\nu^{\text{abs}}\left(\tau_\nu^{\text{abs}} + \tau_\nu^{\text{sca}}\right)} \text{ and } \tau_\nu^{\text{eff}} < \tau_\nu \text{ because } \frac{\tau_\nu^{\text{eff}}}{\tau_\nu} = \sqrt{\frac{\tau_\nu^{\text{abs}}}{\tau_\nu^{\text{abs}} + \tau_\nu^{\text{sca}}}}$$

2.2 Effective optical depth

- The medium is said to be
  - Effectively thin if $\tau_\nu^{\mathrm{eff}} \ll 1$
  - Effectively thick if $\tau_\nu^{\mathrm{eff}} \gg 1$
- If $\tau_\nu^{\mathrm{eff}} \gg 1$, the photons are absorbed and remitted many times before coming out of the medium, and have the time to be in thermal equilibrium with the matter, hence the name “thermalisation length” for l_ν^*.

  The specific intensity emitted by such a layer can be estimated: it is close to that of a non-scattering layer of thickness l_ν^* because all photons emitted by this part of the layer effectively come out (l_ν^* is the distance between creation and destruction)

$$I_\nu = j_\nu^{\mathrm{emi}}\, l_\nu^* = S_\nu^{\mathrm{abs}} \alpha_\nu^{\mathrm{abs}}\, l_\nu^* = S_\nu^{\mathrm{abs}} \sqrt{\epsilon_\nu}$$

  The emerging intensity is attenuated by $\sqrt{\epsilon_\nu}$ with respect to a purely absorbing optically thick layer

2.2 Effective optical depth

- If $\tau_\nu^{\rm eff} \ll 1$, the medium is effectively thin or translucent.

  All photons emitted in the layer end up coming out.

  The emerging intensity is the same as in the case without scattering

- $\tau_\nu^{\rm eff} = \sqrt{\epsilon_\nu}\,\tau_\nu = \sqrt{\tau_\nu^{\rm abs}\left(\tau_\nu^{\rm abs} + \tau_\nu^{\rm sca}\right)}$
  - if $\alpha_\nu^{\rm abs} \gg \alpha_\nu^{\rm sca}$, $\tau_\nu^{\rm eff} \sim \tau_\nu^{\rm abs}$, ie scattering does not play any role
  - If $\alpha_\nu^{\rm abs} \ll \alpha_\nu^{\rm sca}$, $\tau_\nu^{\rm eff} \sim \sqrt{\tau_\nu^{\rm abs} \times \tau_\nu^{\rm sca}}$, the optical depth is higher than in the absence of scattering

2.3 Radiation from a thin medium

- We assume a homogeneous medium, in the sense that temperature, density and extinction coefficient do not depend on the position, but the source function can vary because of scattering. We also assume LTE.

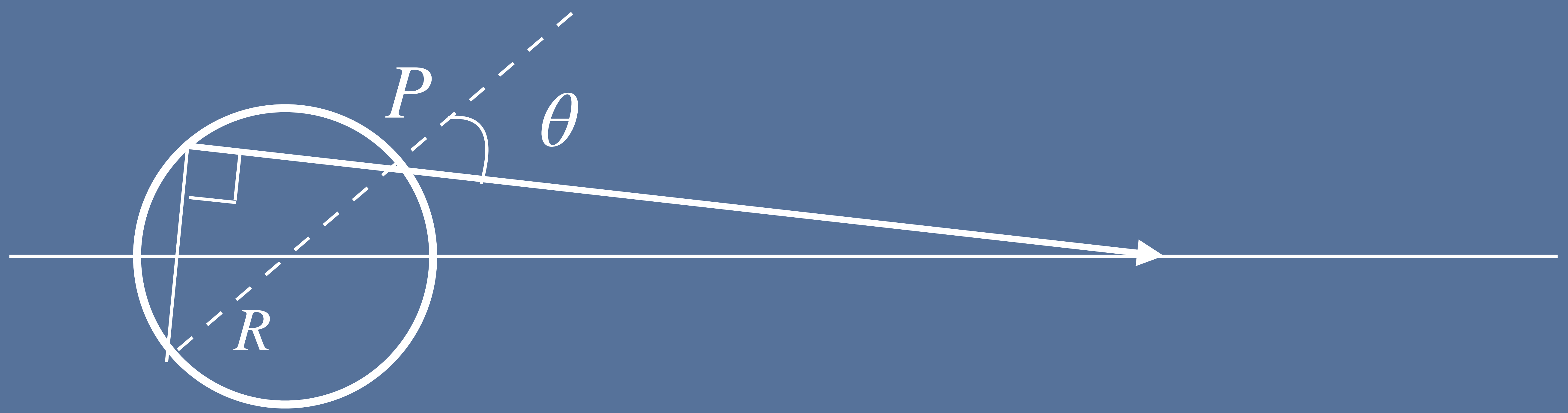
- For an effectively thin medium (ie a medium for which l_ν^* is large so that the photon can cross a very large distance before being absorbed, even if it is scattered), the total monochromatic luminosity is

- $L_\nu = 4\pi\, \alpha_\nu^{\text{abs}}\, B_\nu\, V$, with V the volume of the object

- The term $\alpha_\nu^{\text{abs}}\, B_\nu$ encompasses all thermally created photons that contribute to a beam. By multiplying by $4\pi V$, we obtain the total number of photons that escape the medium, assuming that all photons created anywhere in the medium can leave, independently of the number of scattering events. The information on the direction is lost, therefore we have to use the luminosity

2.3 Radiation from a thin medium

- What is the luminosity of a homogeneous effectively thin sphere with an absorption coefficient $\alpha_\nu^{\rm abs}$ and scattering coefficient $\alpha_\nu^{\rm sca}$?
- We first calculate the luminosity of the sphere assuming there is no scattering, ie for $\alpha_\nu^{\rm sca} = 0$.
- For a point P on the sphere surface, we have $I_\nu(\theta) = \alpha_\nu^{\rm abs} B_\nu\, s = \alpha_\nu^{\rm abs} B_\nu\, 2R\, \cos\theta$, where R is the sphere radius and θ is the angle between the ray and the normal to the surface



2.3 Radiation from a thin medium

- The flux in P is : $F_\nu^+ = 2\pi \int_0^{\frac{\pi}{2}} I_\nu(\theta)\, \cos\theta\, \sin\theta\, d\theta = \frac{4}{3}\pi R\, \alpha_\nu^{\text{abs}}\, B_\nu$
- And the total luminosity : $L_\nu = F_\nu^+\, 4\pi R^2 = 4\pi V \alpha_\nu^{\text{abs}}\, B_\nu$
- This expression of the luminosity is also valid if $\alpha_\nu^{\text{sca}} \neq 0$ because the only difference is the redistribution of the radiation in all directions
- Can a medium be effectively thin and optically thick ?

2.4 Radiation from an optically thick medium

2.4.1 Eddington approximation

- This is also called the diffusion approximation
- In this section, we will treat the monochromatic case (frequency redistribution happens for photon conversion or dust emission, as we will see in the next chapter)
- We use the same method as in Chapter 3, but this time we will use the vector notation.
- The moments of the intensity are written

$$J_\nu = \frac{1}{4\pi} \oint I_\nu(\vec{n})\, d\Omega \quad \vec{H}_\nu = \frac{1}{4\pi} \oint I_\nu(\vec{n})\, \vec{n}\, d\Omega \quad \overline{\overline{K}}_\nu = \frac{1}{4\pi} \oint I_\nu(\vec{n})\, \vec{n}\, \vec{n}\, d\Omega$$

- And the transfer equation: $\vec{n} \cdot \vec{\nabla} I_\nu(\vec{x}, \vec{n}) = j_\nu(\vec{x}) - \alpha_\nu(\vec{x})\, I_\nu(\vec{x}, \vec{n})$

2.4.1 Eddington approximation

- Integrating the equation over the solid angle and dividing by 4π

$$\frac{1}{4\pi}\vec{\nabla}\oint I_\nu(\vec{x},\vec{n})\vec{n}\,d\Omega = \frac{1}{4\pi}\oint j_\nu(\vec{x})\,d\Omega - \alpha_\nu(\vec{x})\frac{1}{4\pi}\oint I_\nu(\vec{x},\vec{n})\,d\Omega$$

which can be written $\vec{\nabla}\,\vec{H}_\nu(\vec{x},\vec{n}) = j_\nu(\vec{x}) - \alpha_\nu(\vec{x})\,J_\nu(\vec{x})$

To write these equations, we have used the fact that α_ν and j_ν do not depend on direction (isotropy)

- Multiplying the equation by $\vec{n}$, integrating the equation over the solid angle and dividing by 4π

$$\frac{1}{4\pi}\vec{\nabla}\oint I_\nu(\vec{x},\vec{n})\vec{n}\,\vec{n}\,d\Omega = \frac{1}{4\pi}\oint j_\nu(\vec{x})\,\vec{n}\,d\Omega - \alpha_\nu(\vec{x})\frac{1}{4\pi}\oint I_\nu(\vec{x},\vec{n})\,\vec{n}\,d\Omega$$

2.4.1 Eddington approximation

Using $\oint \vec{n}\, d\Omega = 0$ we obtain $\vec{\nabla}\,\overline{\overline{K_\nu}} = -\alpha_\nu(\vec{x})\,\overrightarrow{H_\nu}(\vec{x})$

This is a vector equation, ie it represents 3 equations

- Eddington approximation: $K_{ij\nu} = \frac{1}{3}\delta_{ij} J_\nu$
- This is valid for an isotropic radiation field, but we assume it is valid for a quasi-isotropic field (formerly we had $P_\nu = \frac{4\pi}{3c} J_\nu$ and $P_\nu = \frac{4\pi}{c} K_\nu$)
- Using the Eddington approximation, the previous relation becomes

$$\frac{1}{3}\vec{\nabla} J_\nu(\vec{x}) = -\alpha_\nu(\vec{x})\,\overrightarrow{H_\nu}(\vec{x})$$

2.4.1 Eddington approximation

- Combining both equations, we obtain the Eddington equation
$$\frac{1}{3}\vec{\nabla}\left(\frac{1}{\alpha_\nu(\vec{x})}\vec{\nabla}J_\nu(\vec{x})\right)=\alpha_\nu(\vec{x})J_\nu(\vec{x})-j_\nu(\vec{x})$$
This is a 2nd order differential equation.

- It is the diffusion approximation for radiative transfer. The diffusion coefficient is $\dfrac{1}{\alpha_\nu(\vec{x})}$

- From the second moment of the transfer equation, written slightly differently $\vec{F_\nu}(\vec{x}) = -\dfrac{4\pi}{3\,\alpha_\nu(\vec{x})}\vec{\nabla}J_\nu(\vec{x})$, we can see that the radiative flux is proportional to the gradient of the mean intensity, which is exactly what we expect for diffusion.

2.4.1 Eddington approximation

- For thermal emission, we had $S_\nu = \epsilon_\nu\, B_\nu(T) + (1 - \epsilon_\nu)\, J_\nu$
- The diffusion equation is written:

$$\frac{1}{3}\vec{\nabla}\left(\frac{1}{\alpha_\nu}\vec{\nabla}\, J_\nu(\vec{x})\right) = \alpha_\nu\left[J_\nu - S_\nu\right]$$
$$= \alpha_\nu \epsilon_\nu \left[J_\nu - B_\nu(T)\right]$$
$$= \alpha_\nu^{\mathrm{abs}}\left[J_\nu - B_\nu(T)\right]$$

The scattering term has disappeared from the right-hand side of the equation. It remains on the left-hand side, because $\alpha_\nu = \alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{sca}}$

- This equation is only valid in the monochromatic case, ie elastic scattering (the photon is scattered at the same frequency as the incident photon, and thermal emission is also at the same frequency)

2.4.1 Eddington approximation

- We can also derive the relation in the same way as in the case without scattering
- Assuming S_ν isotropic and integrating the transfer equation over $d\Omega$, we obtain the same relation as before: $\dfrac{dF_\nu}{d\tau_\nu} = 4\pi\,(J_\nu - S_\nu)$
- With $d\tau_\nu = \alpha_\nu\, dz = (\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca})\, dz$ and $S_\nu = \dfrac{\alpha_\nu^{\rm abs}\, S_\nu^{\rm abs} + \alpha_\nu^{\rm sca}\, S_\nu^{\rm sca}}{\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm sca}}$, the relation becomes $\dfrac{1}{4\pi}\dfrac{dF_\nu}{dz} = \alpha_\nu\,(J_\nu - S_\nu) = \alpha_\nu J_\nu - \alpha_\nu^{\rm abs} S_\nu^{\rm abs} - \alpha_\nu^{\rm sca} S_\nu^{\rm sca} = \alpha_\nu^{\rm abs}\,(J_\nu - S_\nu^{\rm abs})$

$$\Rightarrow \quad \frac{1}{4\pi}\frac{dF_\nu}{d\tau_\nu^{\rm abs}} = J_\nu - S_\nu^{\rm abs}$$

2.4.1 Eddington approximation

- The second moment of the transfer equation is $c\dfrac{dP_\nu}{d\tau_\nu} = F_\nu$ or $c\dfrac{dP_\nu}{dz} = \alpha_\nu F_\nu$
- With the Eddington approximation: $P_\nu = \dfrac{4\pi}{3c} J_\nu \quad \Rightarrow \quad \dfrac{4\pi}{3}\dfrac{dJ_\nu}{d\tau_\nu} = F_\nu$
- The Eddington equation is therefore $\dfrac{1}{3}\dfrac{d^2 J_\nu}{d\tau_\nu^{\mathrm{eff}\,2}} = J_\nu - S_\nu^{\mathrm{abs}}$

where we have used: $\tau_\nu^{\mathrm{eff}} = \sqrt{\tau_\nu^{\mathrm{abs}}\left(\tau_\nu^{\mathrm{abs}} + \tau_\nu^{\mathrm{sca}}\right)} = \sqrt{\tau_\nu^{\mathrm{abs}}\,\tau_\nu} = \sqrt{\epsilon_\nu}\,\tau_\nu$

2.4.1 Eddington approximation

- The Eddington equation in a scattering medium is identical to the one in a non-scattering medium, but the optical depth is replaced by the effective optical depth.
- The solutions in the non-scattering case can be easily transposed to the scattering case

2.4.2 Relation with the two-beam approximation

- In this approximation, we had $J_\nu = \dfrac{I_\nu^+ + I_\nu^-}{2}$ and $H_\nu = \dfrac{1}{2\sqrt{3}}(I_\nu^+ - I_\nu^-)$
- The formal equations for I^+ and I^- are

$$\frac{1}{\sqrt{3}}\frac{dI^+}{dz} = j - \alpha I^+ = \alpha(S - I^+)$$

$$-\frac{1}{\sqrt{3}}\frac{dI^-}{dz} = j - \alpha I^- = \alpha(S - I^-)$$

2.4.2 Relation with the two-beam approximation

- The half sum and half difference of these equations yield

$$\frac{dH}{dz} = j - \alpha J = \alpha \left(S - J \right) \qquad \frac{1}{\sqrt{3}} \frac{dJ}{dz} = \sqrt{3} \alpha H$$

- Combining both equations we obtain. $\frac{1}{3} \frac{d}{dz} \left(\frac{1}{\alpha} \frac{dJ}{dz} \right) = \alpha \left(J - S \right)$
- We recognise the diffusion equation: the two-beam approximation is therefore mathematically equivalent to the Eddington approximation
- This justifies the choice made $\mu = \pm \frac{1}{\sqrt{3}}$ for the angles of both beams: it is motivated by the necessity to get to the Eddington equation
- We can then use boundary conditions, for example a semi-infinite atmosphere with $I^- = 0$ for $z = 0$
- With these conditions, $H = \frac{1}{2\sqrt{3}}(I^+ - I^-)$ becomes $H = \frac{1}{\sqrt{3}} J$

2.4.3 Rosseland mean opacity

- To determine the variation of the source function with the atmospheric depth within a source (e.g. a star), we need to determine the mean optical depth, which we can link to the geometric depth
- This is also necesary for calculations of inner stellar structures, where we need to know the frequency integrated radiative flux as a function of geometrical depth
- In the deep layers of stars (or similar media), the intensity becomes isotropic and the mean intensity is then equal to the source function, ie $B_\nu(T)$ because LTE is valid.
- With the Eddington approximation $F_\nu = \frac{4\pi}{3}\frac{dJ_\nu}{d\tau_\nu}$

$$F = \int F_\nu \, d\nu = \frac{4\pi}{3}\int \frac{dJ_\nu}{d\tau_\nu} d\nu = \frac{4\pi}{3}\int \frac{dB_\nu}{d\tau_\nu} d\nu = -\frac{4\pi}{3}\int \frac{dB_\nu}{dT}\frac{dT}{dr}\frac{1}{\alpha_\nu} \, d\nu$$

With r the radial coordinate (same as z)

2.4.3 Rosseland mean opacity

$$\Rightarrow \quad F = -\frac{4\pi}{3}\frac{dT}{dr}\frac{1}{\alpha_{\text{ross}}}\int \frac{dB_\nu}{dT}\,d\nu$$

with $\dfrac{1}{\alpha_{\text{ross}}} = \dfrac{\int \frac{1}{\alpha_\nu}\frac{dB_\nu}{dT}\,d\nu}{\int \frac{dB_\nu}{dT}\,d\nu}$, or $\alpha_{\text{ross}} = \dfrac{4\sigma T^3}{\pi}\dfrac{1}{\int \frac{1}{\alpha_\nu}\frac{dB_\nu}{dT}\,d\nu}$

- α_{ross} is the Rosseland mean opacity or Rosseland mean absorption coefficient (harmonic mean of the absorption coefficient). It allows for the calculation of the integrated flux in the deep layers of the atmosphere, as in the “grey” case (ie independent of the frequency)

- It is determined by the parts of the spectrum where the optical depth is small, which is normal because the flux escapes the deep layers essentially at these wavelengths.

2.4.3 Rosseland mean opacity

$$F = -\frac{4\pi}{3}\frac{dT}{dr}\frac{1}{\alpha_{\text{ross}}}\int\frac{dB_\nu}{dT}\,d\nu \quad \text{with } \alpha_{\text{ross}}\,dr = -\,d\tau_{\text{ross}}$$

$$F = \frac{4\pi}{3}\frac{dT}{d\tau_{\text{ross}}}\frac{d}{dT}\int B_\nu\,d\nu = \frac{4\pi}{3}\frac{dT}{d\tau_{\text{ross}}}\frac{d}{dT}\left(\frac{\sigma T^4}{\pi}\right)$$

$$F = \frac{4}{3}\frac{d\left(\sigma T^4\right)}{d\tau_{\text{ross}}}$$

- This is also called “diffusion approximation”, as an analogy with diffusion equations. It is not valid in superficial layers of stars or in optically thin media

2.4.3 Rosseland mean opacity

- It can also be written: $F = -\frac{16}{3}\sigma\frac{T^3}{\alpha_{\text{ross}}}\frac{dT}{dr}$
- The radiative flux depends on the temperature gradient. The radiative energy transport in deep layers of stars is of the same nature as heat conduction, with an effective conductivity $\frac{16\sigma T^3}{3\alpha_{\text{ross}}}$
- The energy flux only depends on one property of the absorption coefficient, its Rosseland mean.
- Because of the $1/\alpha_\nu$ dependency, the frequencies at which the extinction coefficient is small tend to dominate the mean.
- $\frac{dB_\nu}{dT}$ is close to the Planck function, but it peaks for $\frac{h\nu}{kT} \sim 3.8$ instead of 2.8.
- The assumption for this derivation is that all quantities vary slowly with respect to the mean free path

2.4.4 Planck mean opacity

- The emerging intensity due to the emission of an optically thin layer is
- $I_\nu(\tau_\nu, \mu) = I_\nu(0)\, \mathrm{e}^{-\tau_\nu/\mu} + D\, j_\nu/\mu$
- If there is no background intensity: $I_\nu(\tau_\nu, \mu) = D \dfrac{\alpha_\nu}{\mu} S_\nu$
- So: $\displaystyle \int I_\nu\, d\nu = \frac{D}{\mu} \int \alpha_\nu\, S_\nu\, d\nu$

2.4.4 Planck mean opacity

- Assuming LTE additionally, $S_\nu = B_\nu$

$$\Rightarrow \quad \int I_\nu \, d\nu = \frac{D}{\mu} \alpha_P \, \sigma \, T^4 \, ,$$

- with α_P the Planck mean opacity $\quad \alpha_P = \frac{\int \alpha_\nu \, B_\nu \, d\nu}{\int B_\nu \, d\nu}$
- The Planck mean opacity is much less used than the Rosseland mean opacity.
- Contrary to the Rosseland mean opacity, it is weighted by the optically thick parts of the spectrum
- It is a good approximation for outer stellar layers, if they are in radiative equilibrium