Particle-in-cell simulations

Part I: Numerical methods

Benoît Cerutti

CNRS & Université Grenoble Alpes, Grenoble, France.

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Plan of the lectures

- **Wednesday:**

  - *Morning:* The PIC method, numerical schemes and main algorithms.

  - *Afternoon:* Coding practice of the Boris push and the Yee algorithm.

- **Thursday:**

  - *Morning:* Implementation of Zeltron, structure and methods.

  - *Afternoon:* Zeltron hands on relativistic reconnection simulations

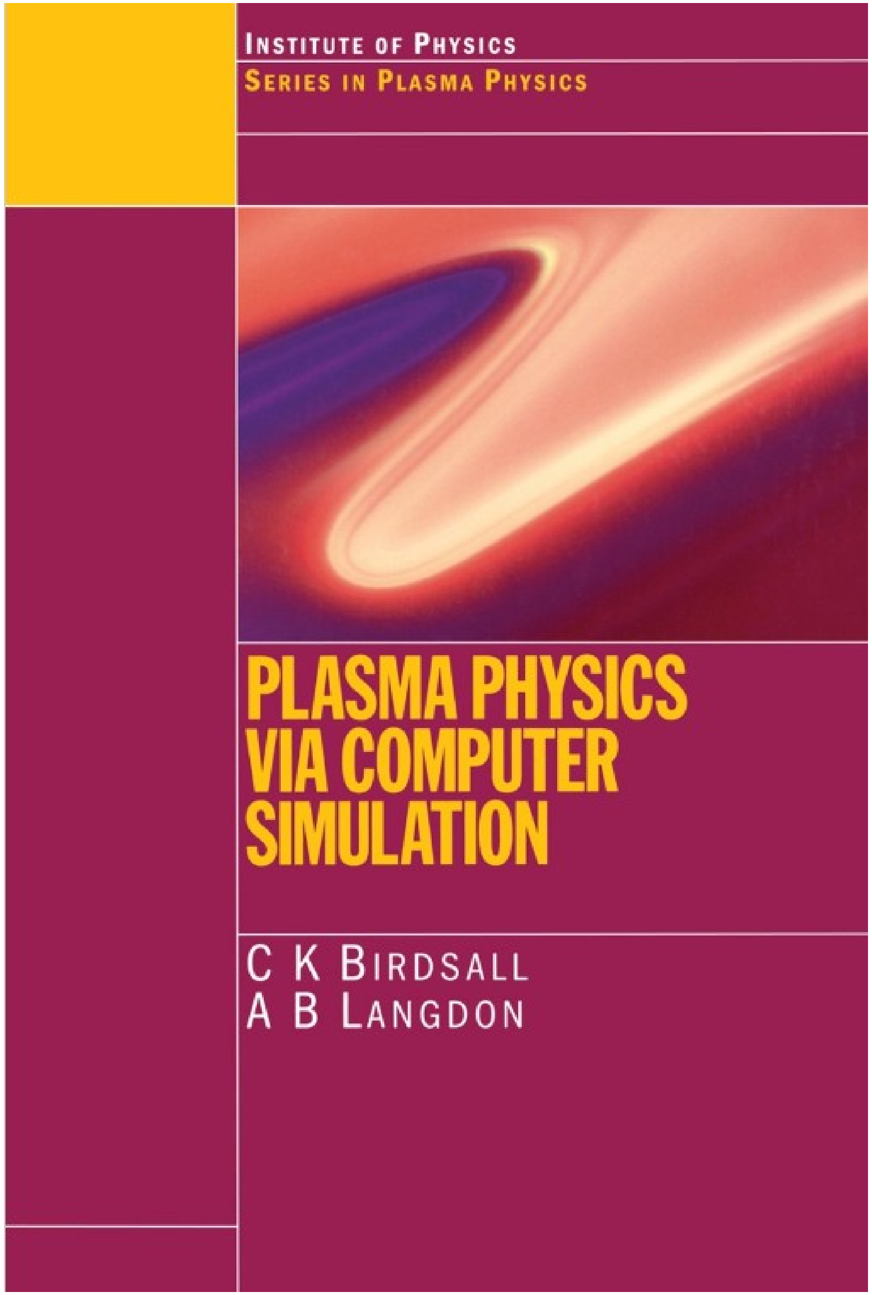
  - *Evening*: Seminar applications of PIC to relativistic magnetospheres.

- ***Friday:***

  - *Morning: B*oundary conditions and parallelization in Zeltron.

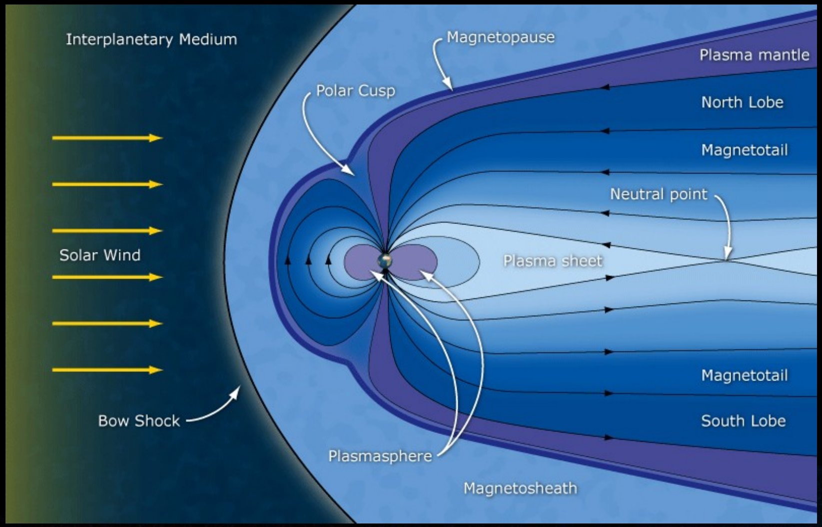
  - *Afternoon:* Zeltron Hands on relativistic collisionless shocks simulations

The Holy book for PIC simulations!



Astrophysical context

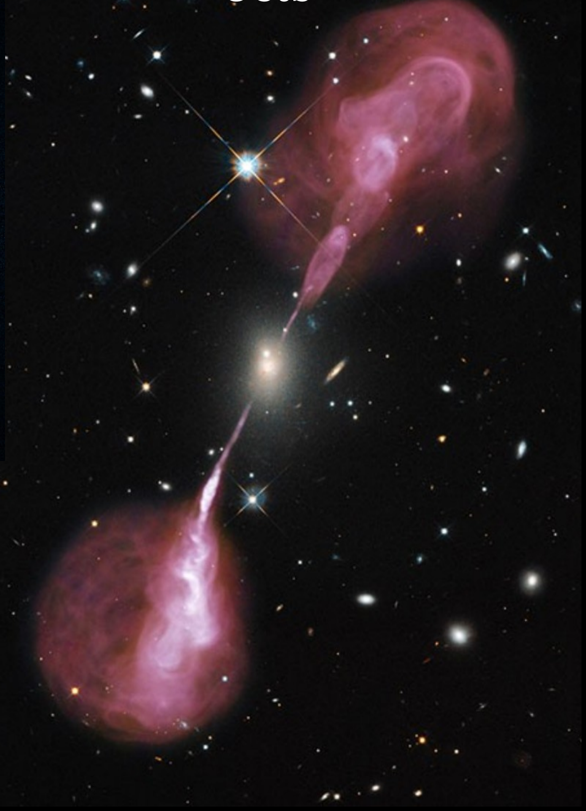
Planetary magnetospheres



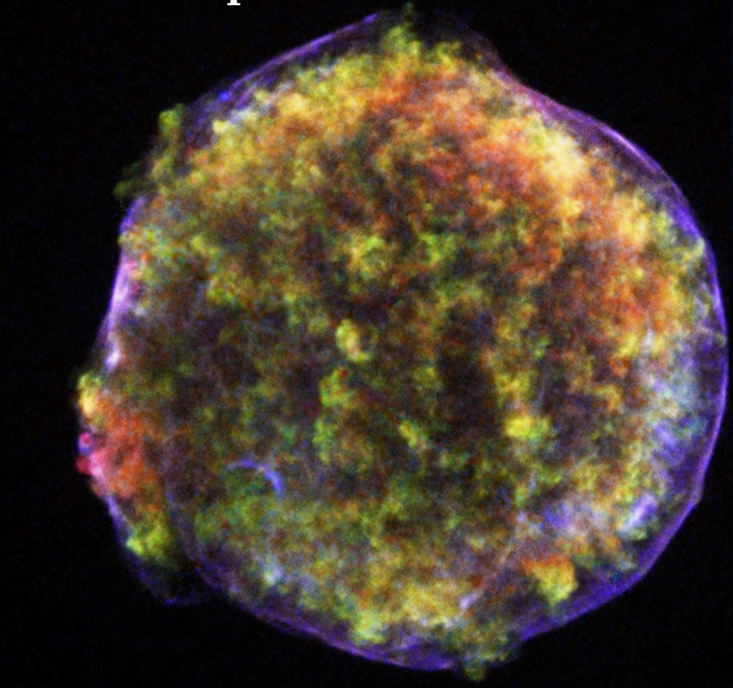
Solar corona & wind, heliosphere

Pulsar Wind Nebulae

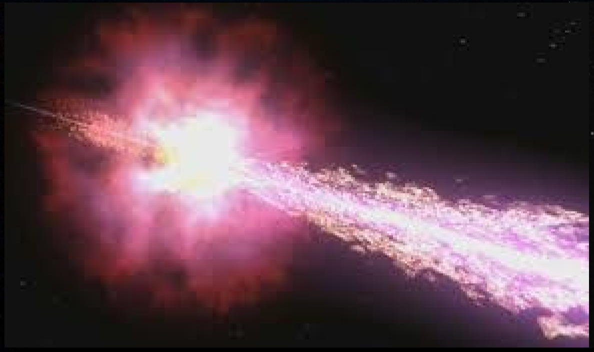
Jets



Supernova Remnants



Gamma-ray bursts



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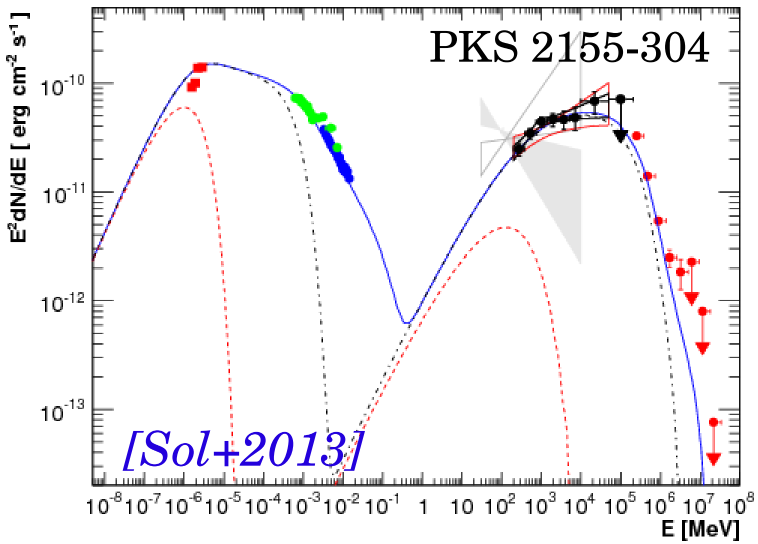
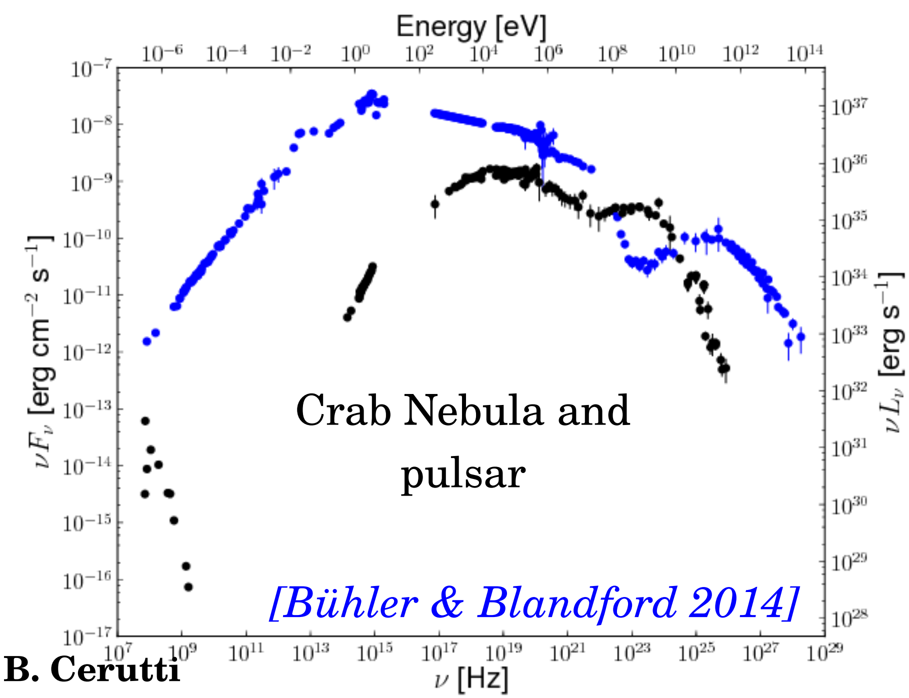
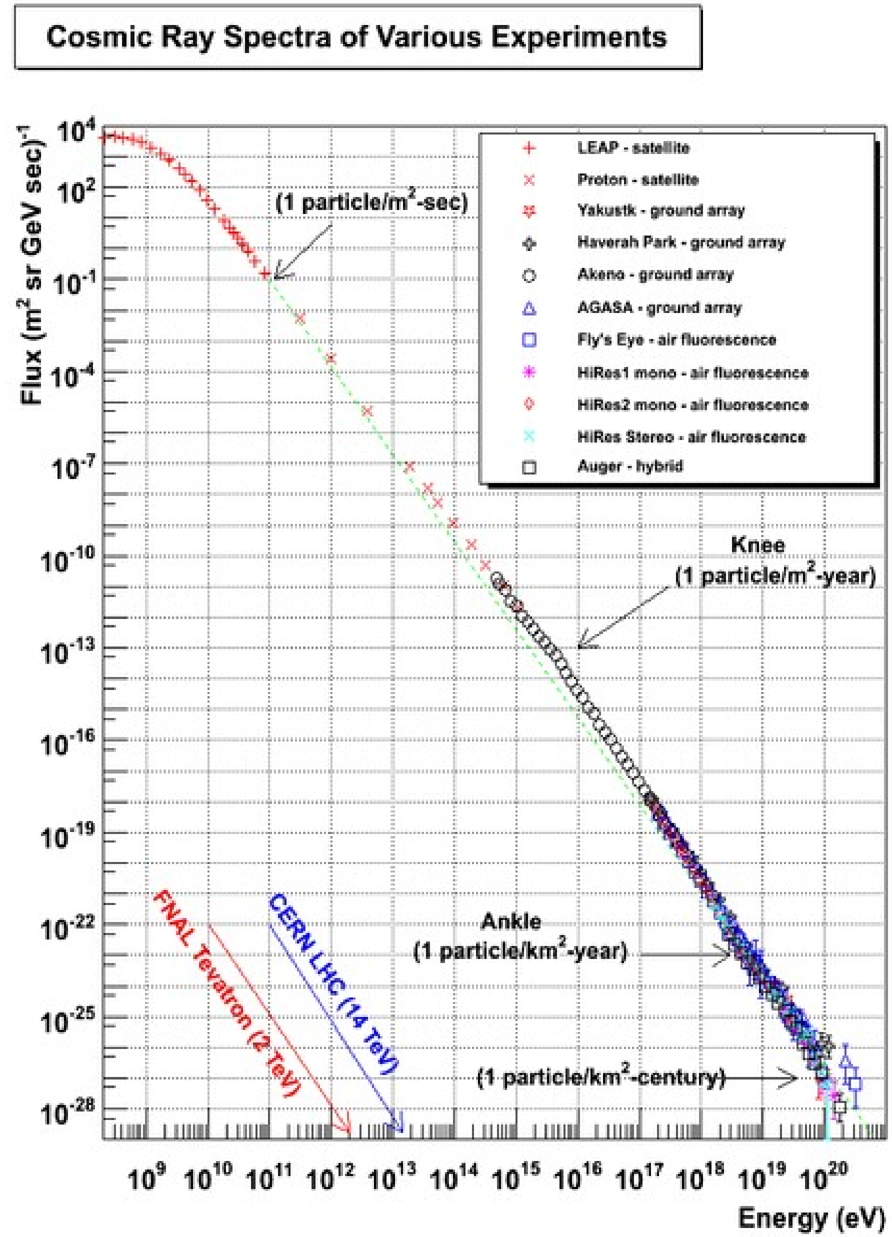
Broad non-thermal distributions

Blazars



Pulsars & Pulsar Wind Nebulae



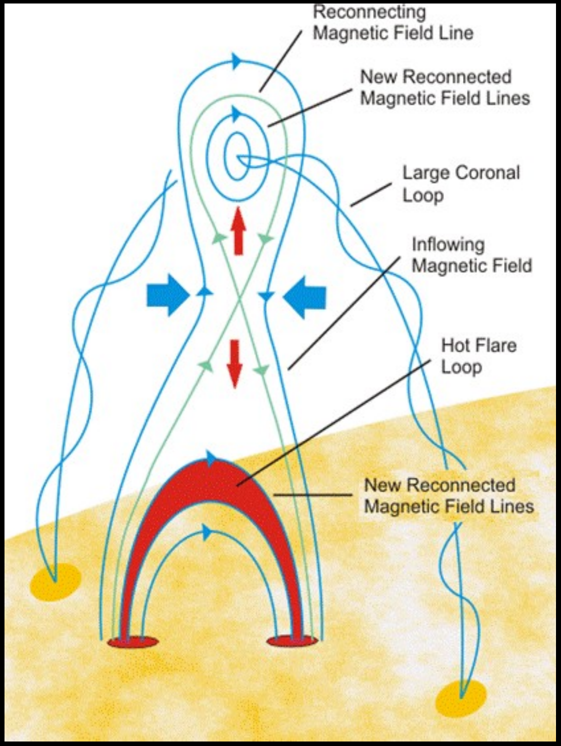


[http://www.physics.utah.edu/~whanlon/spectrum.html]

Particle acceleration processes

Magnetic reconnection

Magnetic energy => Particles

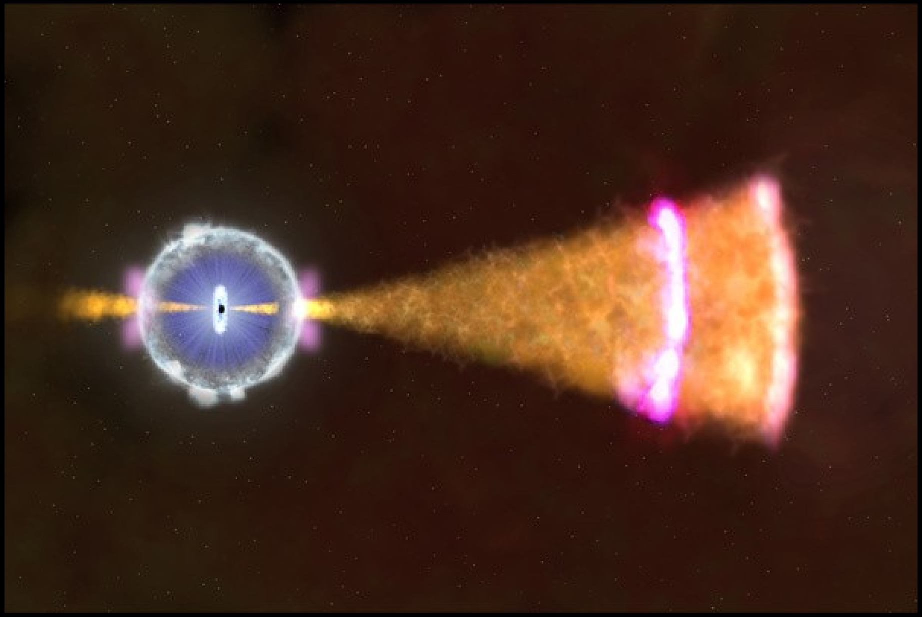


Accretion disk coronae, magnatars, pulsars, jets, GRBs

Hands on session II on Tuesday afternoon

Shocks

Flow kinetic energy => Particles



GRBs, SNRs, PWNe, jets…

Hands on session III on Wednesday afternoon

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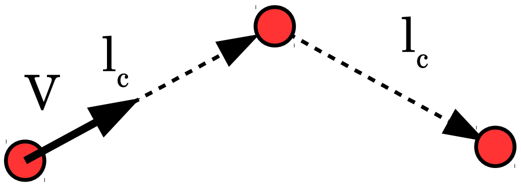
Collisionless plasmas

Collisions thermalizes efficiently the particle distribution, **not good for non-thermal** distributions. In most astrophysical environments, plasmas are **very dilute** so that they are effectively "**collisionless**".

Coulomb collisions **mean free path**: $l_C = \frac{1}{n \sigma_C}$



Frequency of collisions $\nu = \frac{V}{l_C}$

Collisionless plasma if the plasma frequency $\omega_{pe} \gg \nu$

It also implies that there is a large number of particles per **Debye sphere**:

$$N_D = n \lambda_D^3 \gg 1$$

Particles sensitive to **collective plasma phenomena** over binary collisions, particularly important on the **sub-Debye length** and **plasma frequency scales** (plasma frequency and gyroradius).

These microscopic scales are involved in particle acceleration process. **Need to resolve kinetic scales** (≠MHD approach), and system size $L \gg \lambda_D$

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The particle distribution function

Let's start by defining the particle distribution function:

$$f(\boldsymbol{r},\boldsymbol{p},t)=\frac{dN}{d\boldsymbol{r}\,d\boldsymbol{p}}$$

6D in phase space +**1D** in time

The **total number** of particles is given by: $N=\iint_{\boldsymbol{r},\boldsymbol{p}} f(\boldsymbol{r},\boldsymbol{p},t)\,d\boldsymbol{r}\,d\boldsymbol{p}$

The plasma **charge density** by: $\rho=q\int_{\boldsymbol{p}} f(\boldsymbol{r},\boldsymbol{p},t)\,d\boldsymbol{p}$

The plasma **current density** by: $\boldsymbol{J}=q\int_{\boldsymbol{p}} \boldsymbol{v}\,f(\boldsymbol{r},\boldsymbol{p},t)\,d\boldsymbol{p}$

The Vlasov equation

The evolution of distribution function is given by the **Boltzmann equation**:

$$\frac{\partial f}{\partial t}+\frac{\boldsymbol{p}}{\gamma m}\cdot\frac{\partial f}{\partial \boldsymbol{r}}+\boldsymbol{F}\cdot\frac{\partial f}{\partial \boldsymbol{p}}=\left(\frac{\partial f}{\partial t}\right)_{Collisions}$$

For a **collisionless** plasma: $\left(\frac{\partial f}{\partial t}\right)_{Collisions}=0$

And if the fluid feels only the **electromagnetic force:** $\boldsymbol{F}=q\left(\boldsymbol{E}+\frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)$

We obtain the **Vlasov equation**:

$$\frac{\partial f}{\partial t}+\frac{\boldsymbol{p}}{\gamma m}\cdot\frac{\partial f}{\partial \boldsymbol{r}}+q\left(\boldsymbol{E}+\frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial f}{\partial \boldsymbol{p}}=0$$

Along with **Maxwell equations**, we have all equations to model collisionless plasmas.

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Two numerical approaches to solve Vlasov

$$\frac{\partial f}{\partial t}+\frac{\boldsymbol{p}}{\gamma m}\cdot\frac{\partial f}{\partial \boldsymbol{r}}+q\left(\boldsymbol{E}+\frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial f}{\partial \boldsymbol{p}}=0$$

Ab-initio model, no approximations

Directly with a Vlasov-code

Treat phase space as a continuum fluid

Advantages:

- **No noise**, good if tail of f is important dynamically (steep power-law).
- No issue if plasma very **inhomogeneous**.
- **Weak** phenomena can be captured

Limitations:

- Problem **(6+1)D**, hard to fit in the memory, limited resolution.
- Filamentation of the phase space

But becoming more competitive, new development to come, stay tuned!

Not covered here

Indirectctly with a PIC code

Sample phase space with particles

Advantages:

- Conceptually **simple**
- **Robust** and **easy to implement.**
- Easily **scalable** to large number of cores

Limitations:

- **Shot noise**, difficult to sample uniformly f,
- Artificial collisions, requires many particles
- Hard to capture weak/subtle phenomenas
- Load-balancing issues

Main focus of this lecture

The particle approach

The Vlasov equation can be written in the form of **an advection equation:**

$$\frac{\partial f}{\partial t}+\frac{\boldsymbol{p}}{\gamma m}\cdot\frac{\partial f}{\partial \boldsymbol{r}}+q\left(\boldsymbol{E}+\frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial f}{\partial \boldsymbol{p}}=0 \quad \Rightarrow \quad \frac{\partial f}{\partial t}+\nabla\left(f\,\boldsymbol{U}\right)=0$$

Vlasov equation can be solved along **characteristics curves** along which it has the form of a set of ordinary differential equations (the method of characteristics):

$$\frac{d\boldsymbol{p}}{dt}=q\left(\boldsymbol{E}+\frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)$$ Lorentz-Newton equation

$$\frac{d\boldsymbol{r}}{dt}=\boldsymbol{v}$$

The **characteristics curves** corresponds to the **trajectory of individual particles**!

Hence, we can **probe Vlasov equation by solving for the motion of particles**, the larger number, the better!

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The particle approach

The particle approach consists in approximating the distribution function by an ensemble of discrete particles in phase space

Dirac delta function

$$f(\boldsymbol{r},\boldsymbol{p},t) \approx \sum_{k=1}^{N_p} w_k \delta(\boldsymbol{r}-\boldsymbol{r_k}(t))\delta(\boldsymbol{p}-\boldsymbol{p_k}(t))$$

Weight particle k **Position** and **momentum** particle k at time t

It is impossible to have as many particles as real plasmas

=> **Simulation particles are not physical particles.**

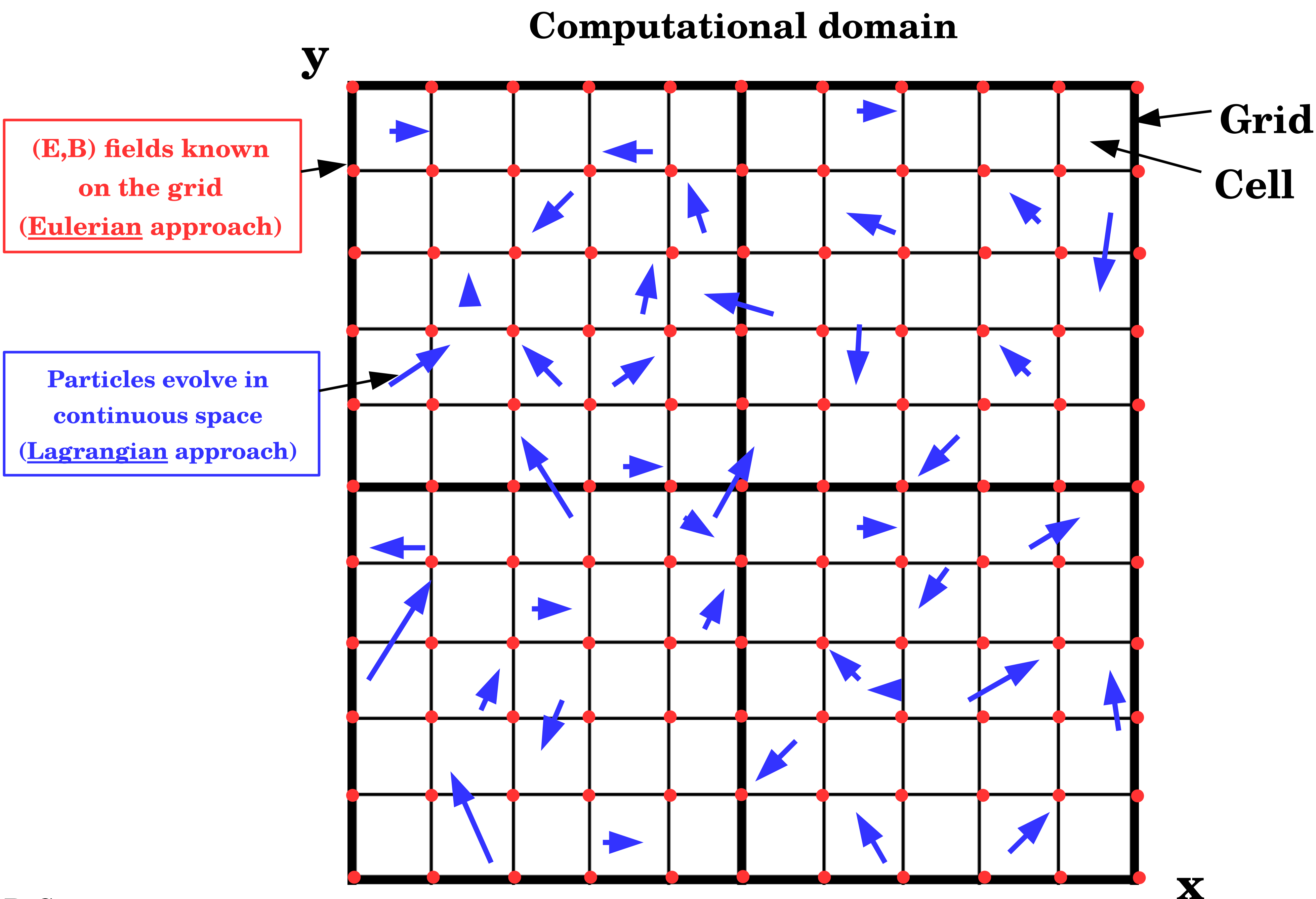
Instead, **they represent a large number of physical particles** which would all follow the same trajectory in phase space, with the same (q/m) ratio.

Simulation particles => "**Macroparticles**"

Then we have:

$$N \approx \sum_{k=1}^{N_p} w_k \qquad \rho \approx \sum_{k=1}^{N_p} q_k w_k \delta(\boldsymbol{r}-\boldsymbol{r_k}(t)) \qquad \boldsymbol{J} = \sum_{k=1}^{N_p} q_k w_k \boldsymbol{v_k} \delta(\boldsymbol{r}-\boldsymbol{r_k}(t))$$

The Particle-In-Cell (PIC) approach



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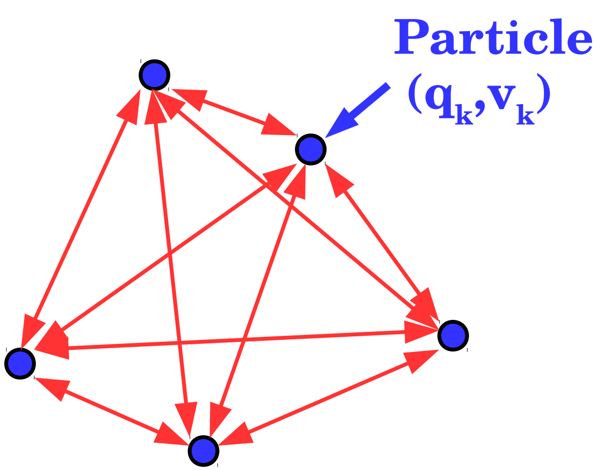
The Particle-In-Cell (PIC) approach

In the PIC approach, the particles do not feel the fields of all the other particles directly. **The particles feel each other through the grid**, via their contribution to the current and charge densities that is deposited on the grid.

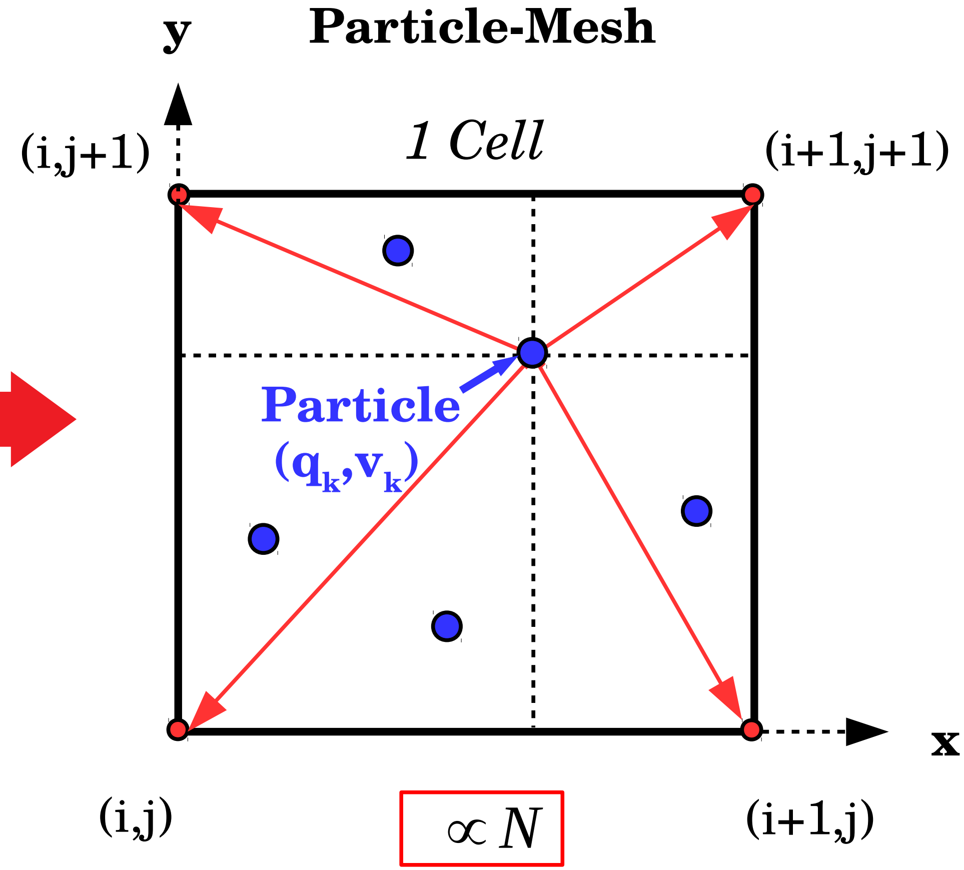
Particle-Particle



Number of pairs: $\frac{N(N-1)}{2} \propto N^2$

Caveats: **Very expansive**, long-range Instantaneous interaction?!

Particle-Mesh



Much cheaper! Propagation of light naturally present via CFL condition

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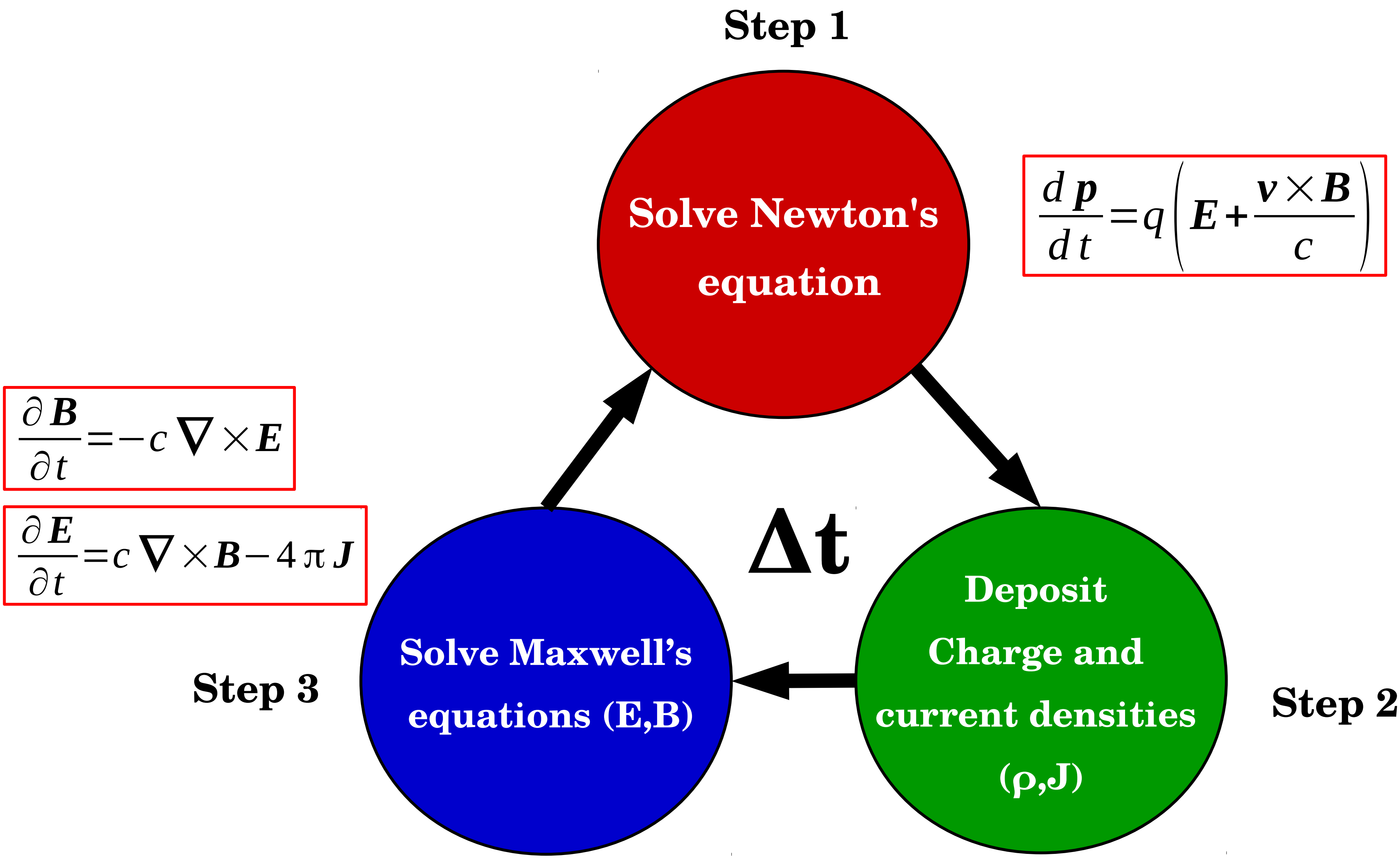
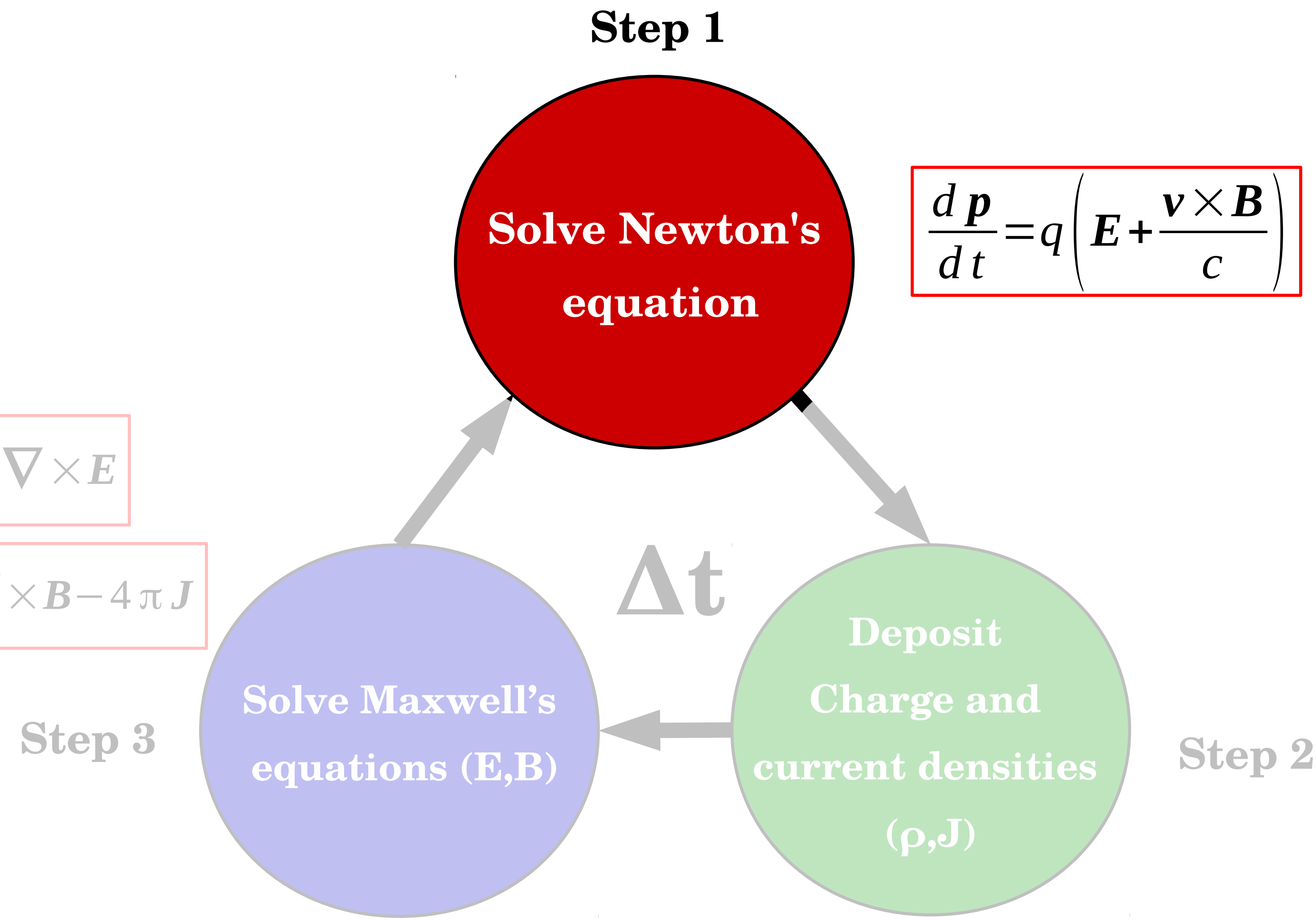
Computation procedure per timestep in PIC



Computation procedure per timestep in PIC



$$\frac{d\boldsymbol{p}}{dt}=q\left(\boldsymbol{E}+\frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)$$

$$\frac{\partial \boldsymbol{B}}{\partial t}=-c\nabla\times\boldsymbol{E}$$

$$\frac{\partial \boldsymbol{E}}{\partial t}=c\nabla\times\boldsymbol{B}-4\pi\boldsymbol{J}$$

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Step 1: Particle push

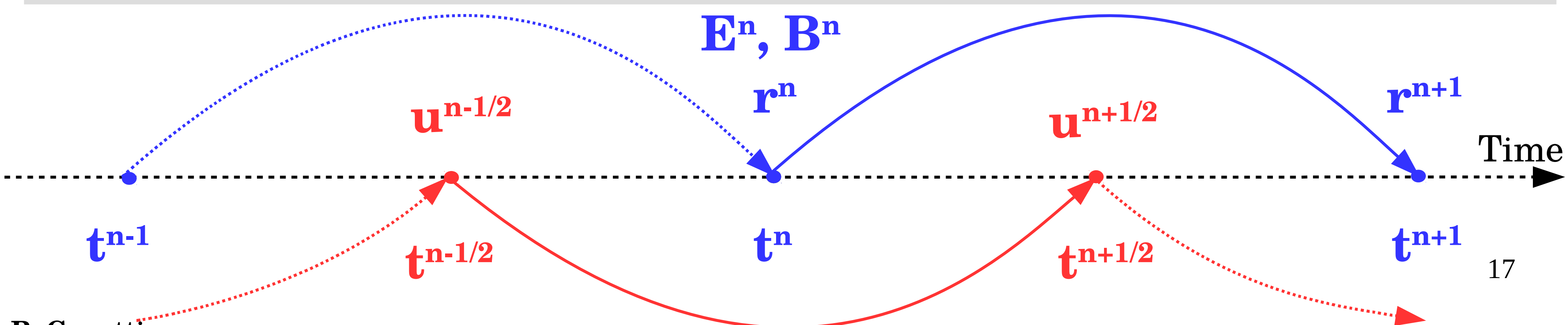
$$\frac{d\boldsymbol{p}}{dt}=q\left(\boldsymbol{E}+\frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right) \quad \Rightarrow \quad \frac{d\boldsymbol{u}}{dt}=\frac{q}{mc}\left(\boldsymbol{E}+\frac{\boldsymbol{u}\times\boldsymbol{B}}{\gamma}\right)$$

$$\frac{d\boldsymbol{r}}{dt}=\frac{c\boldsymbol{u}}{\gamma}$$

Where
$$\begin{cases} \gamma=\dfrac{1}{\sqrt{1-(\boldsymbol{v}/c)^2}} \\ \boldsymbol{u}=\dfrac{\gamma\boldsymbol{v}}{c} \quad \text{(4-velocity)} \end{cases}$$

Explicit **time-centered**, finite-difference scheme (leapfrog integration method):

- **u** and **r** are **staggered in time** by half a time step
- **Second order** accurate but requires only to evaluate function at one time step only (fast and no extra memory needed)
- **Stable** for oscillatory motion (gyromotion) as long as $\Delta t<\Delta t_{CFL}$ (see later)
- Time-reversal and **conserves well energy**
- Implicit methods also exist



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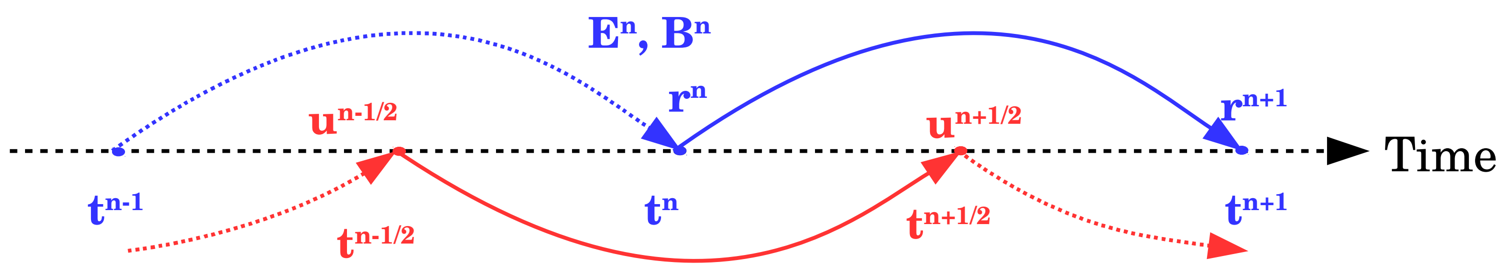
The Boris push (Boris 1970)



$$\frac{d\boldsymbol{u}}{dt} = \frac{q}{mc}\left(\boldsymbol{E} + \frac{\boldsymbol{u}\times\boldsymbol{B}}{\gamma}\right) \Rightarrow \frac{\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^{n-1/2}}{\Delta t} = \frac{q\boldsymbol{E}^n}{mc} + \frac{q}{mc}\left(\frac{\boldsymbol{u}^{n+1/2} + \boldsymbol{u}^{n-1/2}}{2\gamma^n}\right)\times\boldsymbol{B}^n$$

Let's define (Half acceleration):

$$\begin{cases} \boldsymbol{u}^{n+1/2} = \boldsymbol{u}^{+} + \dfrac{q\boldsymbol{E}^n \Delta t}{2mc} \\ \boldsymbol{u}^{n-1/2} = \boldsymbol{u}^{-} - \dfrac{q\boldsymbol{E}^n \Delta t}{2mc} \end{cases}$$

Replacing $\mathbf{u}^+$ and $\mathbf{u}^-$ in Newton's equation gives:

$$\boldsymbol{u}^{+} = \boldsymbol{u}^{-} + \boldsymbol{u}^{-}\times\boldsymbol{s} + \left(\boldsymbol{u}^{-}\times\boldsymbol{w}\right)\times\boldsymbol{s}$$

Where $\boldsymbol{w} = \dfrac{q\boldsymbol{B}^n\Delta t}{2mc\gamma^n}$ and $\boldsymbol{s} = \dfrac{2\boldsymbol{w}}{1+w^2}$

Hands-on I: Code your own Boris push!

More readings: Qin+2013: Why is Boris algorithm so good?

Ripperda+2018: A Comprehensive Comparison of Relativistic Particle Integrators

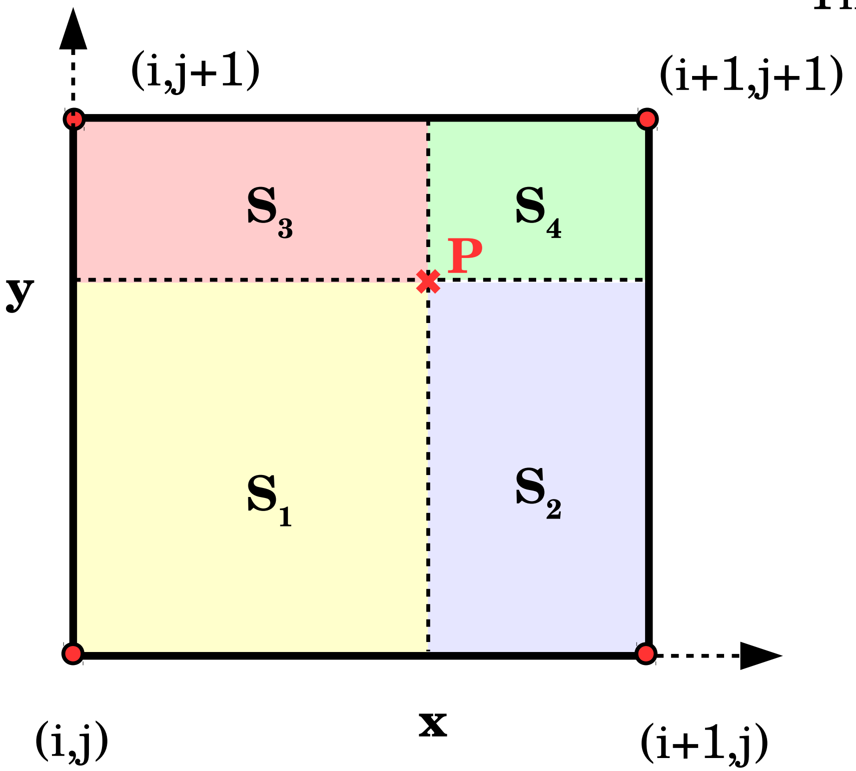
Interpolation of the fields

The fields are known on the mesh only

=> So we need to **interpolate** the fields to the **particle position**

2D Example: Bilinear interpolation ("area weighting", first order)

Consider field F known on the grid nodes **F(i,j)**, and a particle located in **P(x,y)**



Then, the contribution to the field felt by the particle is:

$$W_1 = \frac{S_4}{S_{tot}} F_{i,j} = (1-p)(1-q) F_{i,j}$$

$$W_2 = \frac{S_3}{S_{tot}} F_{i+1,j} = p(1-q) F_{i+1,j}$$

$$W_3 = \frac{S_2}{S_{tot}} F_{i,j+1} = (1-p) q F_{i,j+1}$$

$$W_4 = \frac{S_1}{S_{tot}} F_{i+1,j} = pq F_{i+1,j+1}$$

$$p = (x - x_i)/dx$$

$$q = (y - y_i)/dy$$

$$\boxed{F(x,y) = W_1 + W_2 + W_3 + W_4}$$

… But we can also imagine higher-order scheme.

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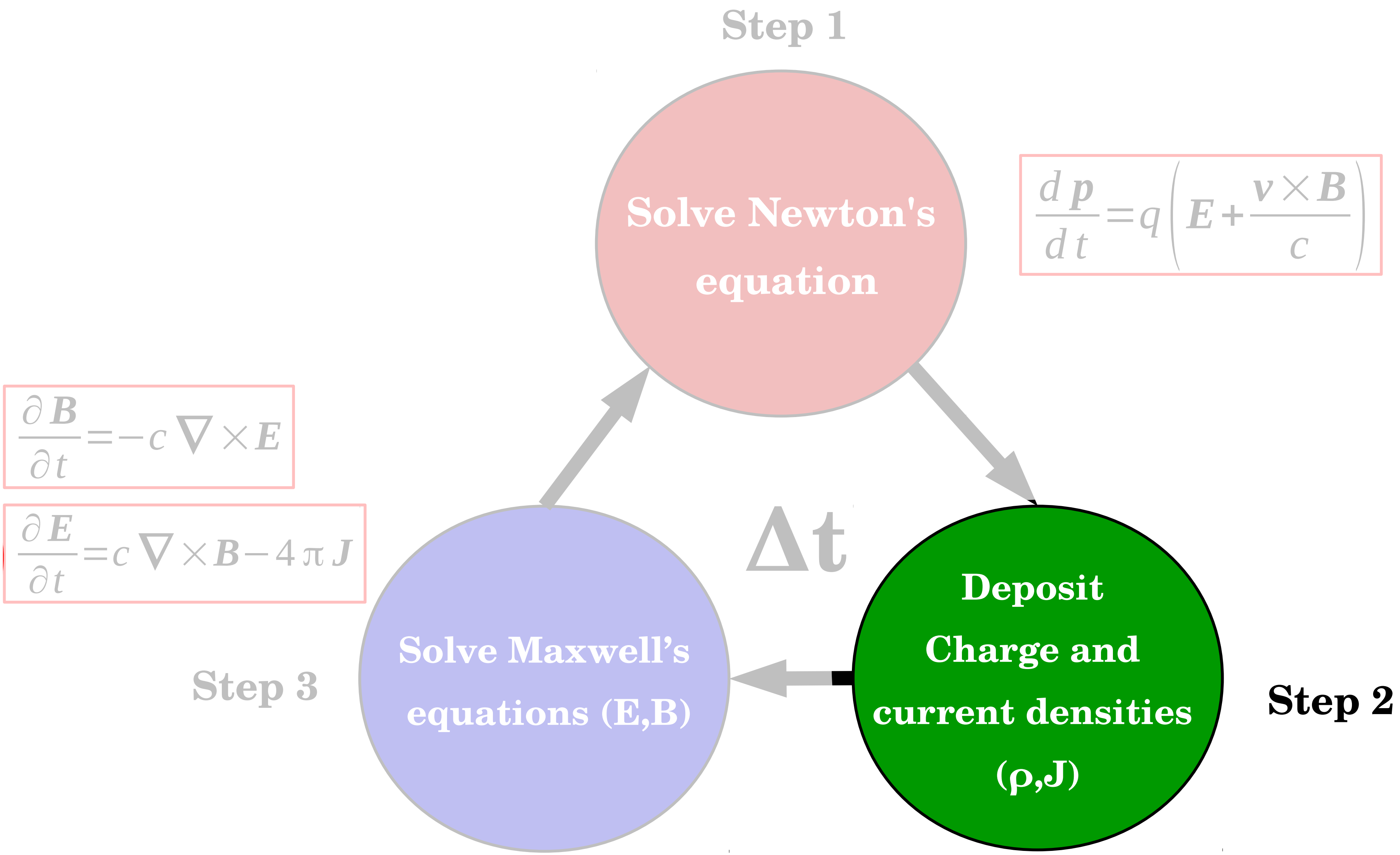
Computation procedure per timestep in PIC



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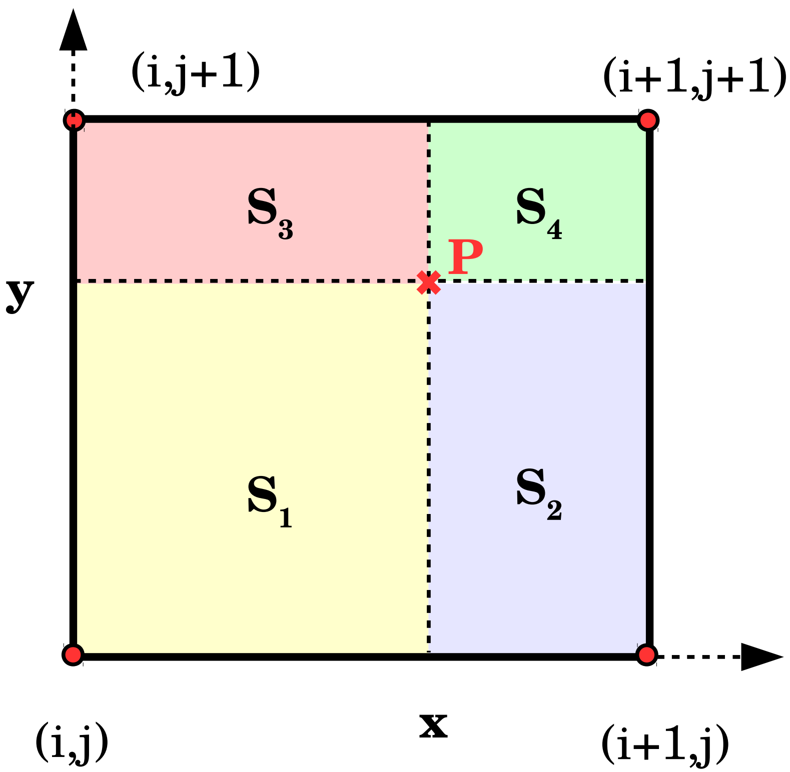
Step 2: Charge and current deposition

In **continuous** space: $\rho \approx \sum_{k=1}^{N_p} q_k w_k \delta\left(\boldsymbol{r}-\boldsymbol{r}_{\boldsymbol{k}}(t)\right) \qquad \boldsymbol{J} = \sum_{k=1}^{N_p} q_k w_k \boldsymbol{v}_{\boldsymbol{k}} \delta\left(\boldsymbol{r}-\boldsymbol{r}_{\boldsymbol{k}}(t)\right)$

On the **grid**: $\rho_{i,j} \approx \sum_{k=1}^{N_{cell}} q_k w_k S\left(\boldsymbol{r}-\boldsymbol{r}_{\boldsymbol{k}}(t)\right)$, where S is a **"shape" function**

2D Example: Bilinear interpolation ("area weighting", first order)



Then, the contributions of all particles to the current is:

$$\boldsymbol{J}_{i,j} = \sum_{k=1}^{N_{cell}} q_k w_k \left(1-p_k\right)\left(1-q_k\right) \boldsymbol{v}_k$$

$$\boldsymbol{J}_{i+1,j} = \sum_{k=1}^{N_{cell}} q_k w_k p_k \left(1-q_k\right) \boldsymbol{v}_k$$

$$\boldsymbol{J}_{i,j+1} = \sum_{k=1}^{N_{cell}} q_k w_k \left(1-p_k\right) q_k \boldsymbol{v}_k$$

$$\boldsymbol{J}_{i+1,j+1} = \sum_{k=1}^{N_{cell}} q_k w_k p_k q_k \boldsymbol{v}_k$$

$$p_k = \left(x_k - x_i\right)/dx$$

$$q_k = \left(y_k - y_i\right)/dy$$

Even though the particles are point-like, they have an **effective size** that is felt through the deposition of currents on the grid. In this case, their effective shape is triangular.

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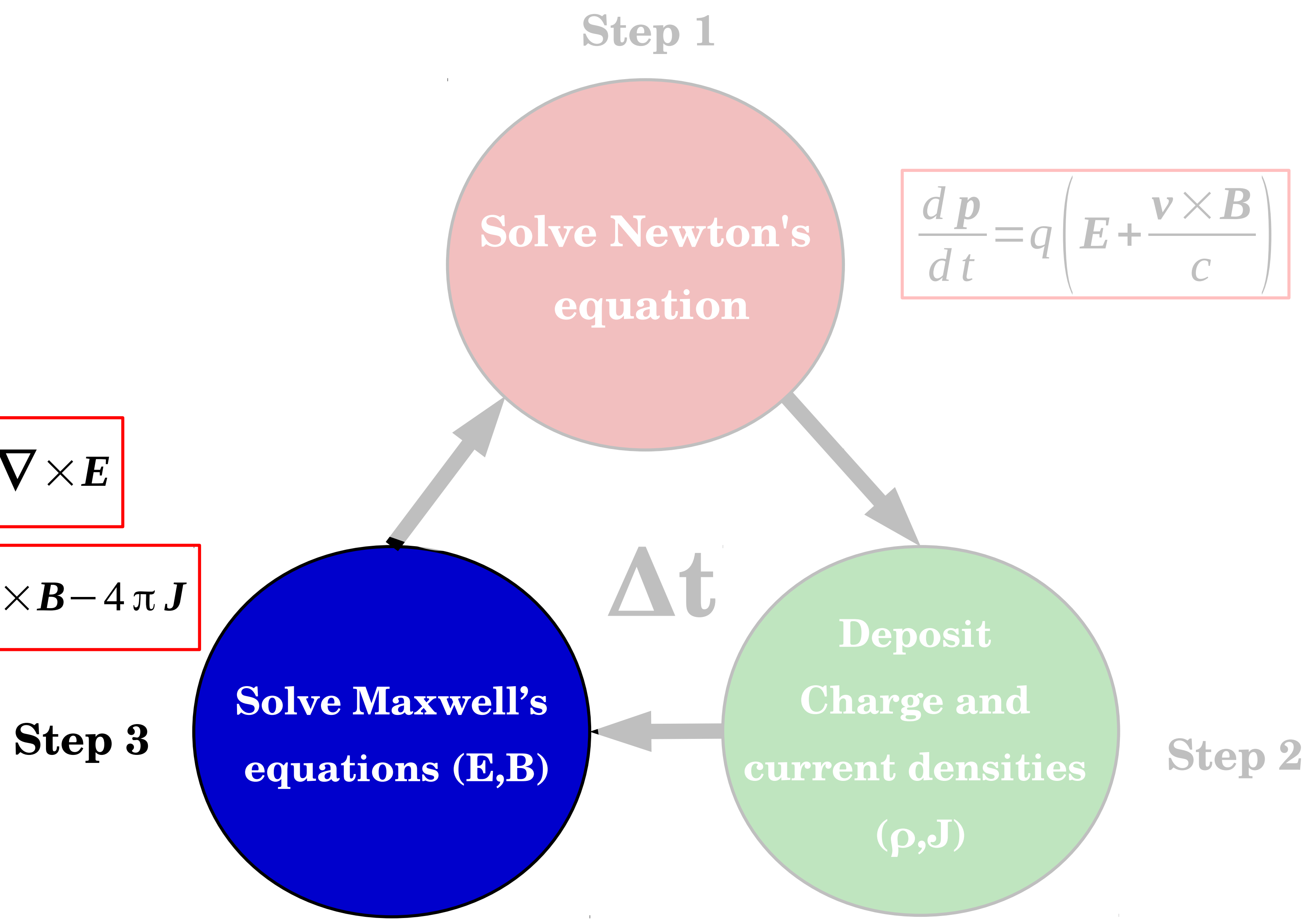
Computation procedure per timestep in PIC

$$\frac{\partial \mathbf{B}}{\partial t} = -c\,\nabla \times \mathbf{E}$$

$$\frac{\partial \mathbf{E}}{\partial t} = c\,\nabla \times \mathbf{B} - 4\pi \mathbf{J}$$



$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right)$$

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Step 3: Maxwell equations

In Gaussian cgs units:

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \qquad \nabla \cdot \boldsymbol{B} = 0$$

$$\frac{\partial \boldsymbol{E}}{\partial t} = c\nabla \times \boldsymbol{B} - 4\pi \boldsymbol{J} \qquad \frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E}$$

In principle, need to solve for the **time-dependent equations only**, then the other two should be **automatically satisfied**, but this is not necessarily true due to **truncation errors**.

The total particle charge is conserved, but not necessarily the charge deposited on the grid!

$$\nabla \cdot \boldsymbol{E} \neq 4\pi\rho$$

Option 1: Correct the E field and solve Poisson equation

Option 2: Parabolic/Hyperbolic divergence cleaning *[Marder 1987, Munz+2000]*

Option 3: Charge conserving deposition scheme *[Esirkepov 2001, Villasenor & Buneman 1992]*

$$\nabla \cdot \boldsymbol{B} = 0$$

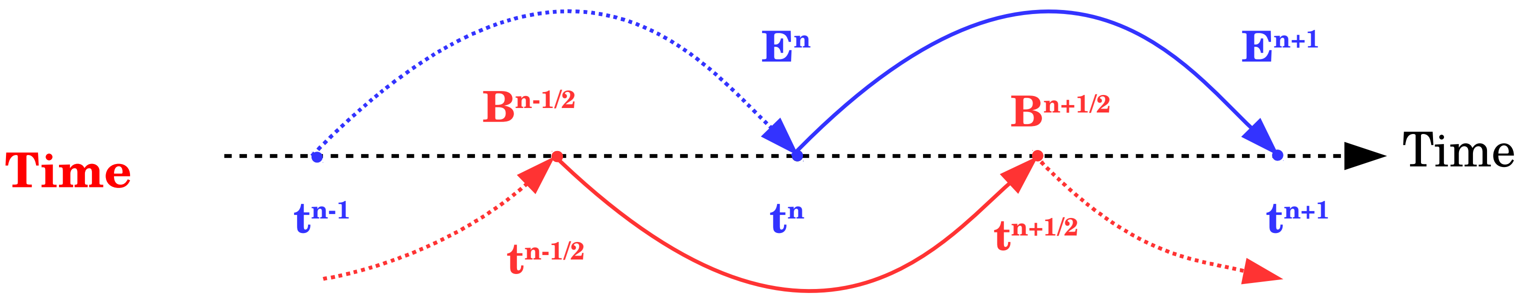
Automatically satisfied to machine roundoff precision with the Yee Algorithm! *[Yee 1966]*

Yee algorithm

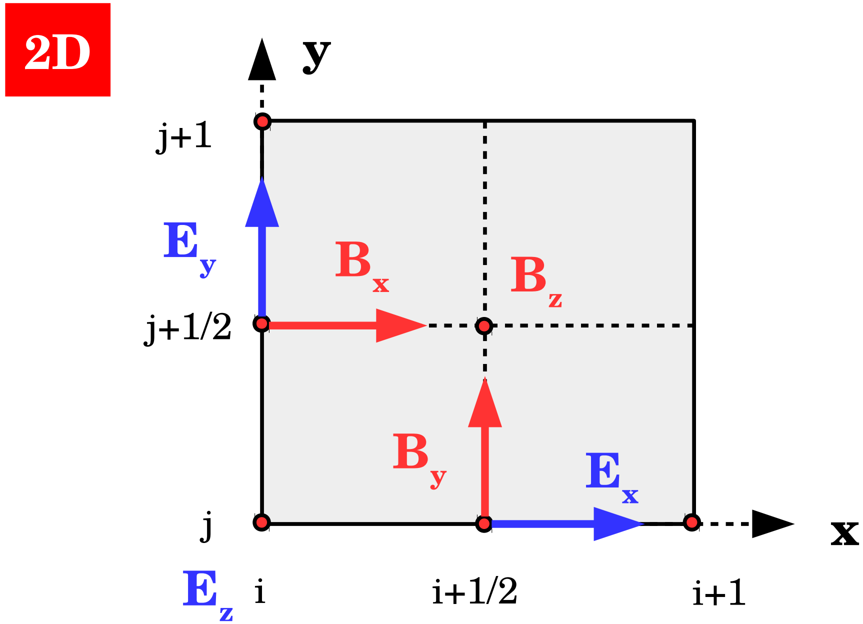
$$\frac{\partial \boldsymbol{E}}{\partial t} = c\nabla\times\boldsymbol{B} - 4\pi\boldsymbol{J}$$

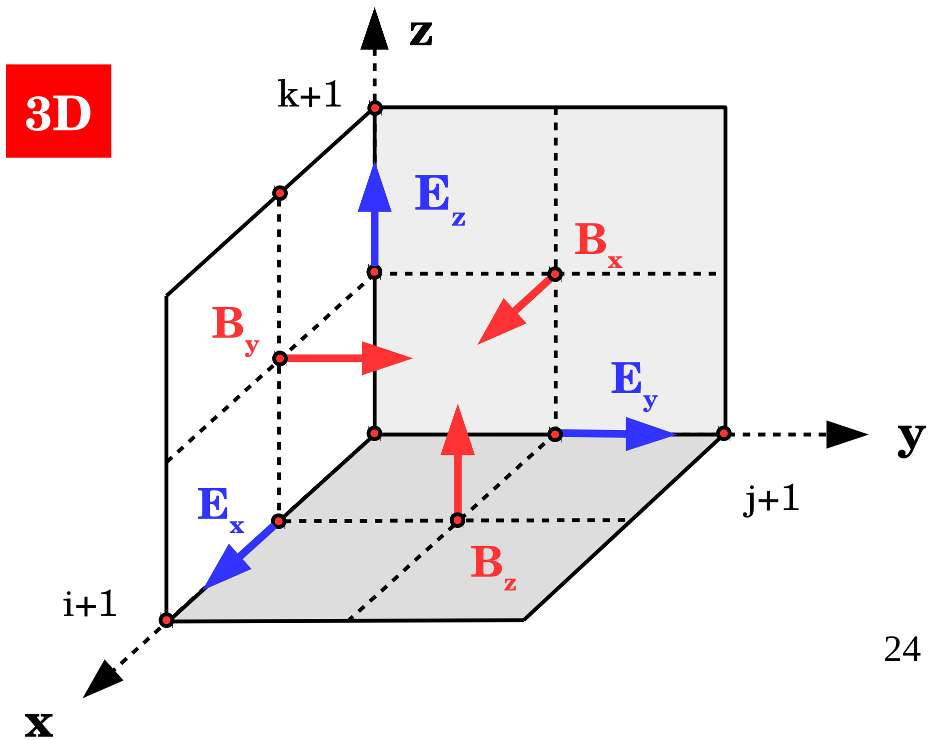
$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla\times\boldsymbol{E}$$

The fields are **staggered in both space and in time!**



Space: staggered mesh (**"Yee mesh"**)





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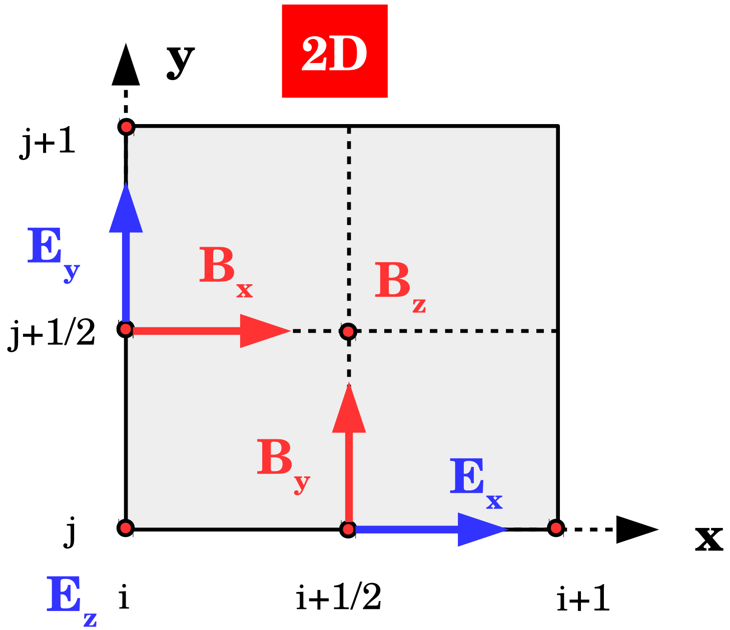
Yee algorithm

Finite-Difference Time-Domain (FDTD) scheme: 2nd in space and time

Hands-on I: Code your own Yee solver!



Explicit components in 2D + vacuum

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla\times\boldsymbol{E} \qquad \frac{\partial \boldsymbol{E}}{\partial t} = c\nabla\times\boldsymbol{B}$$

$$\frac{(E_x)^{n+1}_{i+1/2,j}-(E_x)^{n}_{i+1/2,j}}{\Delta t} = c\frac{(B_z)^{n+1/2}_{i+1/2,j+1/2}-(B_z)^{n+1/2}_{i+1/2,j-1/2}}{\Delta y}$$

$$\frac{(E_y)^{n+1}_{i,j+1/2}-(E_y)^{n}_{i,j+1/2}}{\Delta t} = -c\frac{(B_z)^{n+1/2}_{i+1/2,j+1/2}-(B_z)^{n+1/2}_{i-1/2,j+1/2}}{\Delta x}$$

$$\frac{(B_z)^{n+1/2}_{i+1/2,j+1/2}-(B_z)^{n-1/2}_{i+1/2,j+1/2}}{\Delta t} = -c\frac{(E_y)^{n}_{i+1,j+1/2}-(E_y)^{n}_{i,j+1/2}}{\Delta x}+\frac{(E_x)^{n}_{i+1/2,j+1}-(E_x)^{n}_{i+1/2,j}}{\Delta y}$$

Very **robust** and **stable** if the **Courant-Friedrichs-Lewy** (CFL) condition is fulfilled:

1D: $\left(\frac{c\Delta t}{\Delta x}\right)^2 < 1$ **2D:** $(c\Delta t)^2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) < 1$ **3D:** $(c\Delta t)^2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}\right) < 1$

Physics: The **Debye length** and the **plasma frequency** must be resolved in PIC

$$\frac{\Delta x}{\Lambda_D} < 1 \qquad \omega_{pe}\Delta t < 1$$

Numerical dispersion of the Yee solver

We are looking for plane waves solutions

$$(F)_{i,j}^{n} = F_0 \exp I\left(n\omega t - ik_x \Delta x - jk_y \Delta y\right)$$

$$\left(\partial_t E_x\right)_{i+1/2,j}^{n+1/2} = \frac{2I\left(E_x\right)_{i+1/2,j}^{n+1/2}}{\Delta t} \sin\frac{\omega \Delta t}{2}$$

$$\left(\partial_y E_x\right)_{i+1/2,j+1/2}^{n} = \frac{2I\left(E_x\right)_{i+1/2,j+1/2}^{n}}{\Delta y} \sin\frac{\omega \Delta y}{2}$$

Dispersion relation

$$\left[\frac{1}{c\Delta t}\sin\left(\frac{\omega \Delta t}{2}\right)\right]^2 = \left[\frac{1}{\Delta x}\sin\left(\frac{k_x \Delta x}{2}\right)\right]^2 + \left[\frac{1}{\Delta y}\sin\left(\frac{k_y \Delta y}{2}\right)\right]^2$$

Instead of:

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2$$

FDTD leads to numerical dispersion (last lecture)

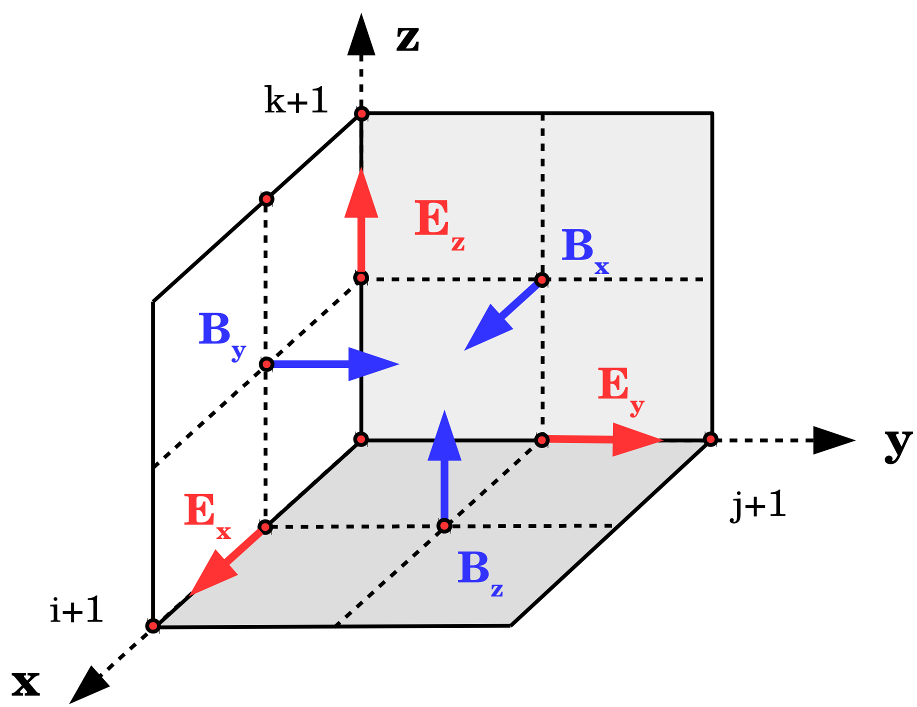
Beyond the standard electromagnetic PIC code

Non-Cartesian grid

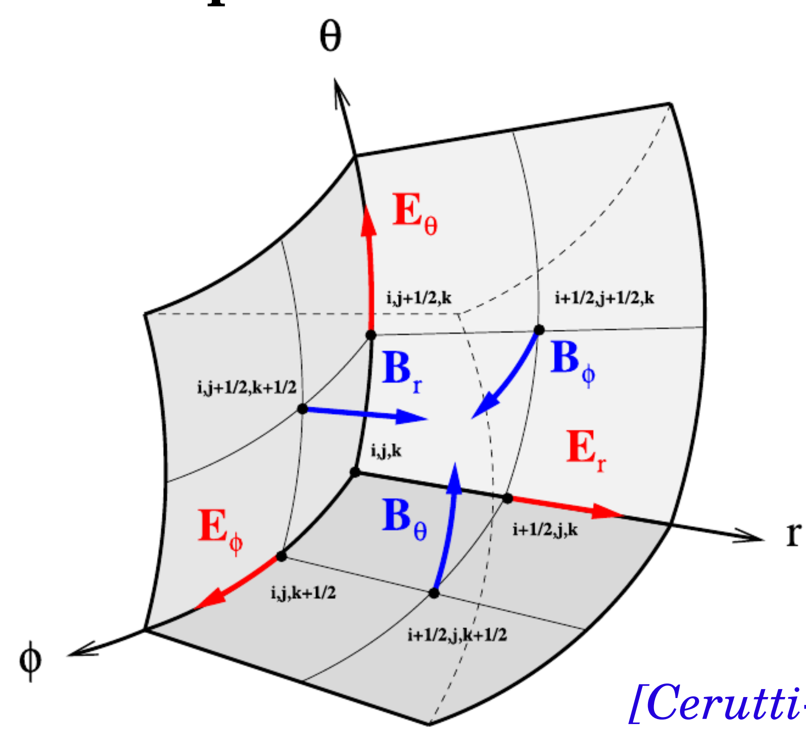
Sometimes, it can be more interesting to use **non-cartesian** grid to take advantage of the symmetries of the system.

=> Simplifies the initial setup load balancing and boundary conditions

Cartesian Yee-mesh



Spherical Yee-mesh



[Cerutti+15,16]

Applications to plasmas around a central object.

Examples: pulsar magnetospheres, accreting systems (see tomorrow's seminar)

Beyond the standard electromagnetic PIC code

Emission of non-thermal radiation

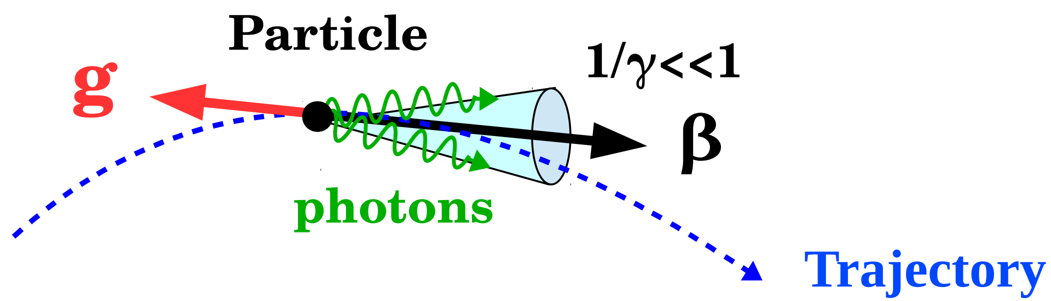
The frequency of the energetic radiation is often not resolved by the grid!

Example: Synchrotron radiation critical frequency: $\omega_{syn} \propto \gamma^2 (qB/mc) = \gamma^3 \omega_c \gg 1/\Delta t$

Hence, **photons must be added as a separate species.**

Also, the radiation reaction force must be added in the equation of motion explicitly:

$$\frac{d\boldsymbol{p}}{dt} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right) + \boldsymbol{g}$$



The radiation reaction force is then given by the **Landau-Lifshitz formula** (classical electrodynamics):

$$\boldsymbol{g} \approx \frac{2}{3} r_e^2 \left[\left(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B} \right) \times \boldsymbol{B} + \left(\boldsymbol{\beta} \cdot \boldsymbol{E} \right) \boldsymbol{E} \right] - \frac{2}{3} r_e^2 \gamma^2 \left[\left(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B} \right)^2 - \left(\boldsymbol{\beta} \cdot \boldsymbol{E} \right)^2 \right] \boldsymbol{\beta}$$

For **inverse Compton** scattering (isotropic external source in the Thomson regime):

$$\boldsymbol{g} = -\frac{4}{3} \sigma_T \gamma^2 U_{rad} \boldsymbol{\beta}$$

Applications to e.g., PWN, AGN jets

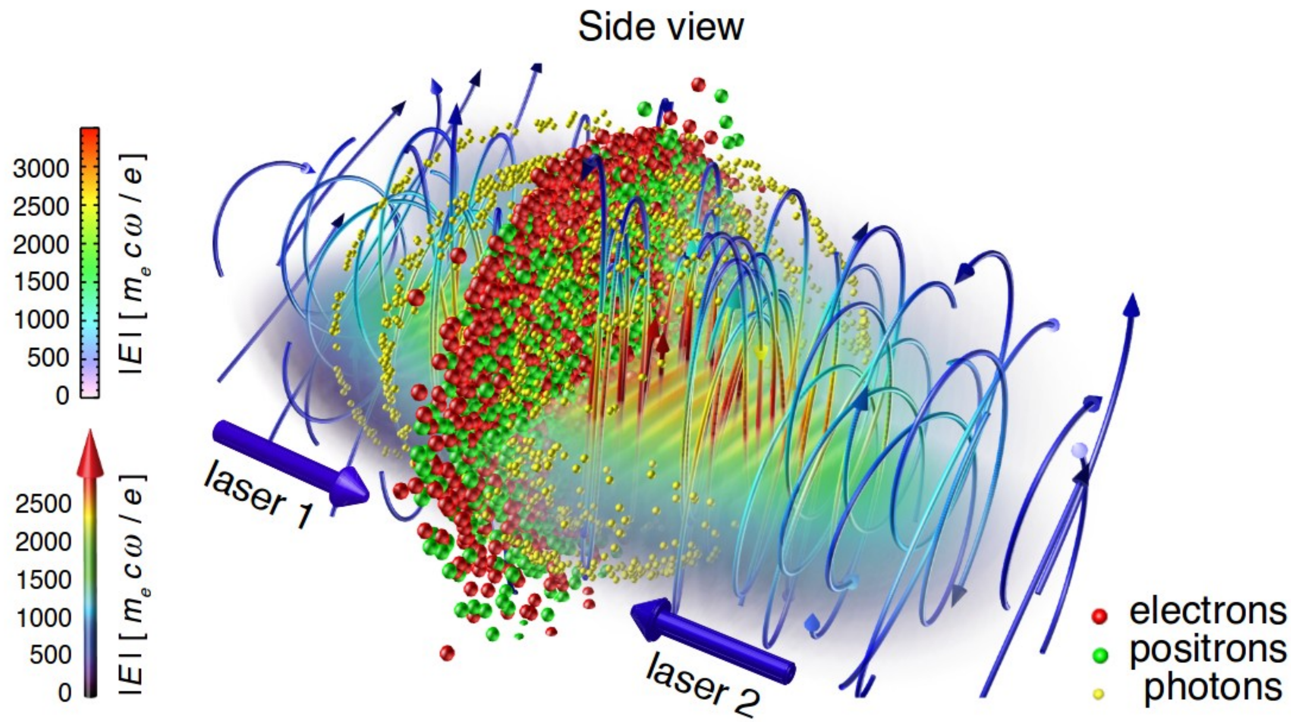
[See Cerutti+2013, 2016]

Beyond the standard electromagnetic PIC code

Pair creation, QED effects

The laser-plasma community is adding extra physics for the next generation of **high-intensity laser** that will reach a fraction of the **critical field** => **QED** effects and **pair creation** important

$$E_{QED} = \frac{m_e^2 c^3}{e\hbar} \approx 4.4 \times 10^{13} \quad G$$



Grismayer+2015

Regime relevant to **pulsars**, **magnetars** ($B>B_{QED}$), and **black hole** magnetospheres.

PIC with pair creation start being used in astrophysics: *Timokhin 2010, Chen & Beloborodov 2014, Philippov + 2015a,b.*

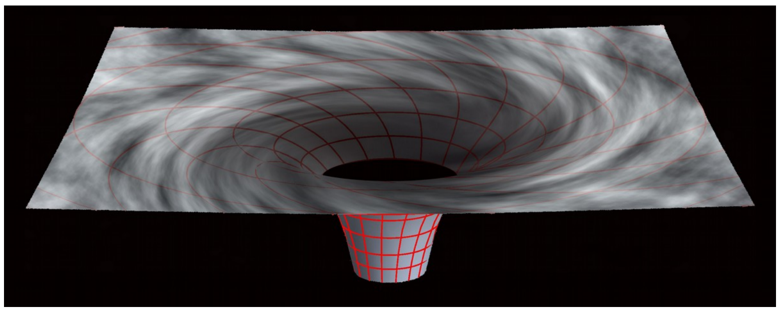
B. Cerutti

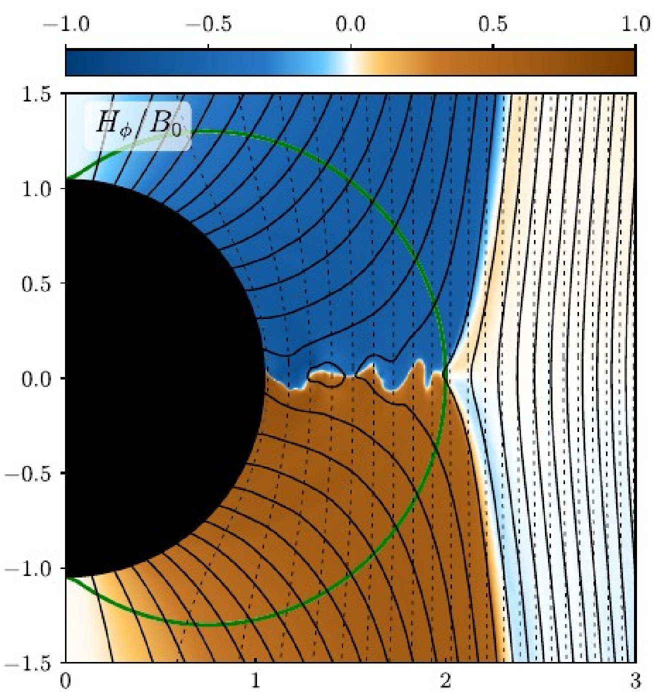
Beyond the standard electromagnetic PIC code

Non-Euclidian metric

Application to e.g., **black hole** magnetospheres and **pulsars**.

“3+1” space-time foliation: Equations are solved on local inertial frames (“FIDO” observers)





Metric term

Maxwell:

$$\frac{1}{\sqrt{\gamma}}\frac{\partial\left(\sqrt{\gamma}\boldsymbol{B}\right)}{\partial t}=-c\,\nabla\times\boldsymbol{E}$$

$$\frac{1}{\sqrt{\gamma}}\frac{\partial\left(\sqrt{\gamma}\boldsymbol{D}\right)}{\partial t}=c\,\nabla\times\boldsymbol{H}-4\pi\boldsymbol{J}$$

Equation of motion:

$$\frac{\mathrm{d}x^i}{\mathrm{d}t}=v^i=\frac{\alpha}{\Gamma}\gamma^{ij}u_j-\beta^i,$$

$$\frac{\mathrm{d}u_i}{\mathrm{d}t}=-\Gamma\partial_i\alpha+u_j\partial_i\beta^j-\frac{\alpha}{2\Gamma}\partial_i(\gamma^{lm})u_l u_m+\frac{\alpha}{m}\mathcal{L}_i$$

Metric induced terms

 Parfrey, Philippov & Cerutti (2019)

A few words about hybrid PIC codes

An important limitation of full PIC methods is the **limited separation of scales.** Only microscopic systems can be modelled.

In particular, it's hard to model electron/ions plasmas with realistic mass ratio
Plasma frequency $\omega_p \propto 1/\sqrt{m} \rightarrow \omega_{pe}/\omega_{pi} = \sqrt{m_i/m_e} \approx 43$

Hence, **ion acceleration is hard to capture with PIC** (except in the ultra-relativistic limit).

Hybrid code: *[e.g., see Winske+2003]*
Ions are **PIC** particles: $m_i \frac{d\boldsymbol{v}_i}{dt} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}_i \times \boldsymbol{B}}{c}\right)$

Electrons are treated as a massless neutralizing **fluid** (method works for **non-relativistic plasmas**): $n_e m_e \frac{d\boldsymbol{V}_e}{dt} = 0 = -e n_e q\left(\boldsymbol{E} + \frac{\boldsymbol{V}_e \times \boldsymbol{B}}{c}\right) - \boldsymbol{\nabla} \cdot P_e$

Example: Application to non-relativistic shock acceleration. *[Gargaté & Spitkovsky 2011, Caprioli & Spitkovsky 2014]*

Summary Part I

- PIC methods appropriate to model particle acceleration in **relativistic collisionless** outflows.
- Main algorithms for explicit PIC codes:
  - Evolving particles: **Boris push**
  - Evolving the fields: **FDTD Yee method**
- **PIC** is very **robust**, **scalable**, and **versatile** to various setup.

<u>**A few useful references:**</u>

- C.K. Birdsall, A.B Langdon, "*Plasma Physics via Computer Simulation*", Series in Plasma Physics
- R.W. Hockney, J.W. Eastwood, "*Computer Simulation Using Particles*"
- Philip L. Pritchett, "*Particle-in-Cell Simulation of Plasmas – A Tutorial*", J. Büchner, C.T. Dum, M. Scholer (Eds.): LNP 615, pp. 1–24, 2003.
- J. Büchner, "*Vlasov-code simulation*", Advanced Methods for Space Simulations, edited by H. Usui and Y. Omura, pp. 23–46, 2007.