# New theoretical approaches to black holes

Eric Gourgoulhon

Laboratoire de l'Univers et de ses Théories (LUTH)
CNRS / Observatoire de Paris / Université Paris Diderot
F-92195 Meudon, France

eric.gourgoulhon@obspm.fr

http://www.luth.obspm.fr/~luthier/gourgoulhon/

*based on a collaboration with*

José Luis Jaramillo

Instituto de Astrofísica de Andalucía, Granada, Spain

**Jean-Pierre Lasota, X-ray binaries, accretion disks and compact stars**
Trzebieszowice Castle, Lądek Zdrój, Poland, 7-13 October 2007

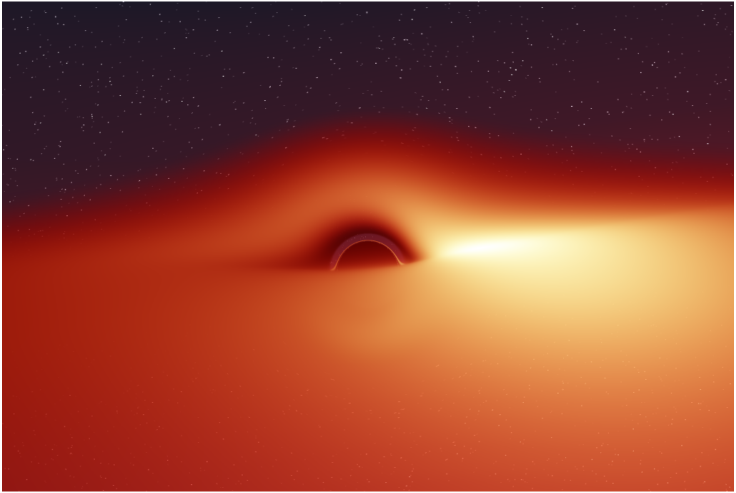
# Plan

1. Review of "classical" black holes
2. New approaches to black holes
3. Geometry of hypersurface foliations by spacelike 2-surfaces
4. A Navier-Stokes-like equation
5. Area evolution and energy equation

# Outline

# What is a black hole ?

... for the astrophysicist: a very deep gravitational potential well



[J.A. Marck, CQG **13**, 393 (1996)]

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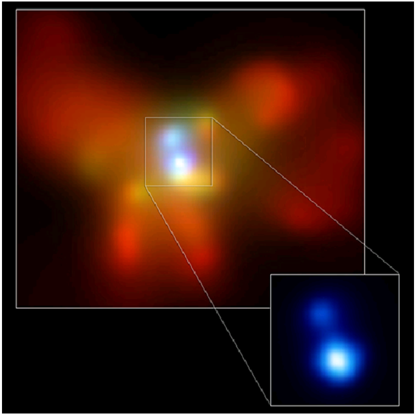
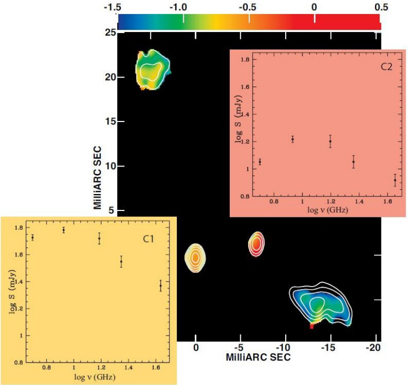


Binary BH in galaxy NGC 6240

$d = 1.4$ kpc

[Komossa et al., ApJ **582**, L15 (2003)]




Binary BH in radio galaxy 0402+379

$d = 7.3$ pc

[Rodriguez et al., ApJ **646**, 49 (2006)]

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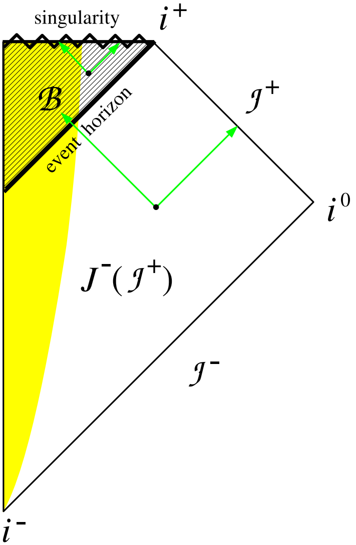


... for the mathematical physicist:

$$\boxed{\mathcal{B} := \mathscr{M} - J^-(\mathscr{I}^+)}$$

i.e. the region of spacetime where light rays cannot escape to infinity

- $\mathscr{M}$ = asymptotically flat manifold
- $\mathscr{I}^+$ = future null infinity
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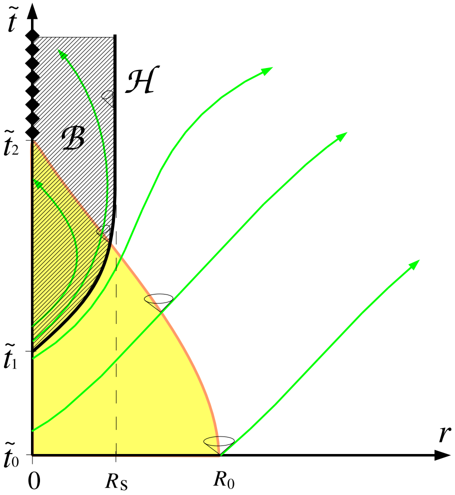


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## Black Holes in an Expanding Universe

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**The question of gravitational collapse to a black hole is treated in the context of an expanding universe.**

THE recent extensive discussions[1] of the problem of black holes and their creation are based on the assumption that the space-time is asymptotically flat. This condition is not satisfied in an expanding Friedmann model such as is generally accepted as a model describing quite satisfactorily the large-scale structure of our Universe. It is not important in the present epoch of the

$$4\pi\rho = \frac{m'}{r'\,r^2} \tag{4}$$

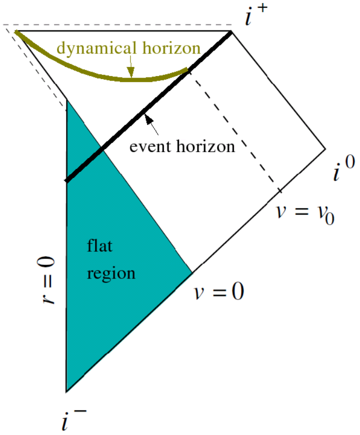
here $E$ $(1+2E>0)$ and $m$ are functions of $R$ only, $\rho$ denotes the density of matter, $'$ and $\cdot$ are shorthands for the derivatives with respect to $R$ and $t$.

It is interesting that equation (3) has exactly the same form as the Newtonian energy equation, but now $m(R)$ denotes the effective gravitational mass and not the "proper" invariant mass $M(R) = \int \rho \sqrt{-g}\, d^3x$.

In order to obtain a unique solution of equations (2) to (4) one should specify the initial conditions assigning, for example, at the time $t=t_0(R)$, the values of $r(R,t_0(R)), \dot{r}(R,t_0(R)), m(R)$,

# Even when applicable, this definition is highly non-local !

The determination of the boundary of $J^-(\mathscr{I}^+)$ requires the knowledge of the entire future null infinity. Moreover this is not locally linked with the notion of strong gravitational field:




[Ashtekar & Krishnan, LRR **7**, 10 (2004)]

Example of event horizon in a **flat** region of spacetime:

Vaidya metric, describing incoming radiation from infinity:

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dv\,dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

with

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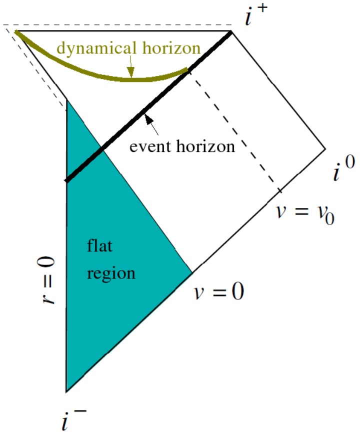


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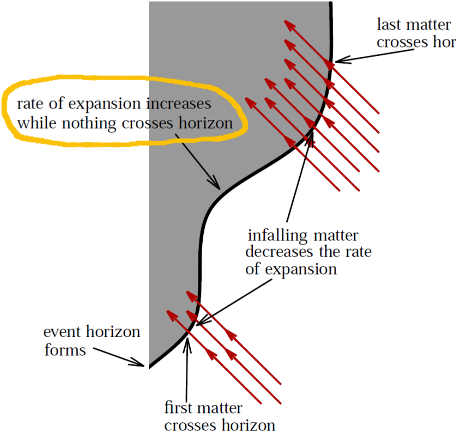
$\Rightarrow$ no local physical experiment whatsoever can locate the event horizon

# Another non-local feature: teleological nature of event horizons

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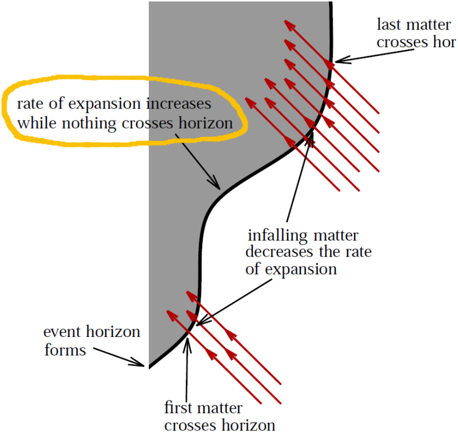


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To deal with black holes as physical objects, a local definition would be desirable

# Outline

# Local characterizations of black holes

Recently a **new paradigm** appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- trapping horizons (Hayward 1994)
- isolated horizons (Ashtekar et al. 1999)
- dynamical horizons (Ashtekar and Krishnan 2002)
- slowly evolving horizons (Booth and Fairhurst 2004)

All these concepts are **local** and are based on the notion of trapped surfaces

*Motivations:* quantum gravity, numerical relativity

# What is a trapped surface ?

## 1/ Expansion of a surface along a normal vector field


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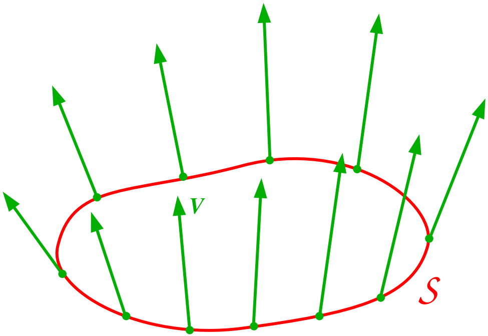


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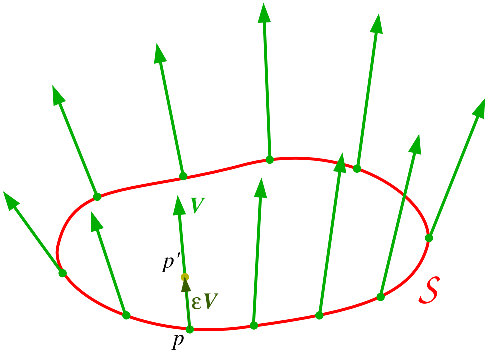


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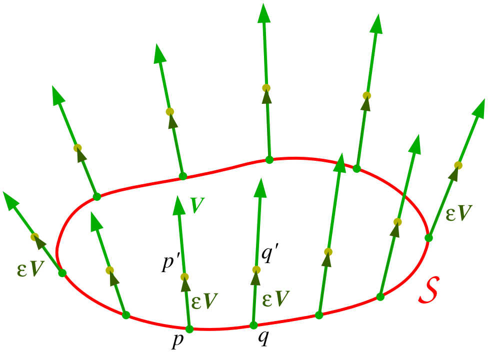


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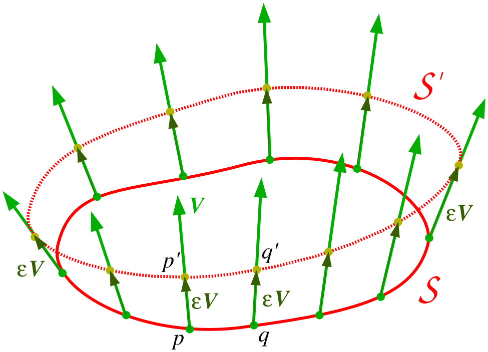


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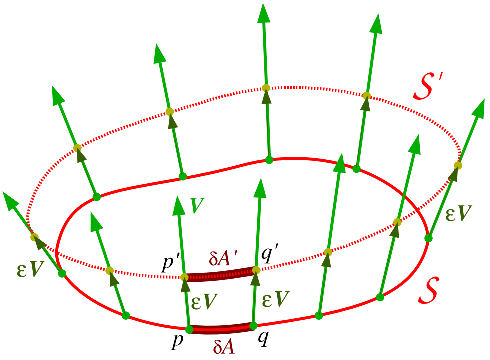


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At each point, the **expansion of $\mathcal{S}$ along $\boldsymbol{v}$** is defined from the relative change in the area element $\delta A$:

$$\boxed{\theta^{(\boldsymbol{v})} := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\frac{\delta A' - \delta A}{\delta A} = \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \ln\sqrt{q} = q^{\mu\nu}\nabla_\mu v_\nu}$$

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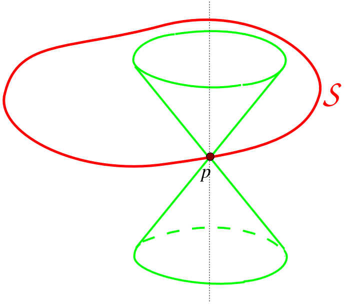


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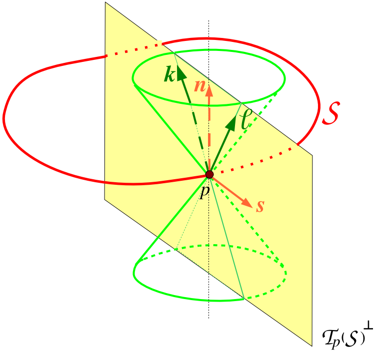


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$\exists$ two future-directed null directions orthogonal to $\mathcal{S}$:

$\boldsymbol{\ell}$ = outgoing, expansion $\theta^{(\boldsymbol{\ell})}$

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In flat space, $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} > 0$

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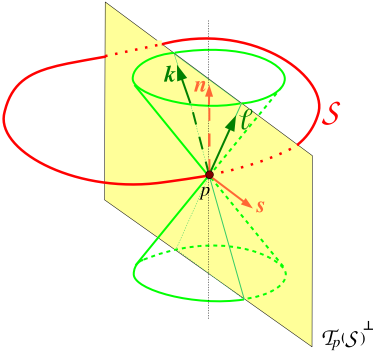


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$\mathcal{S}$ is trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} < 0$ [Penrose 1965]

$\mathcal{S}$ is marginally trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$

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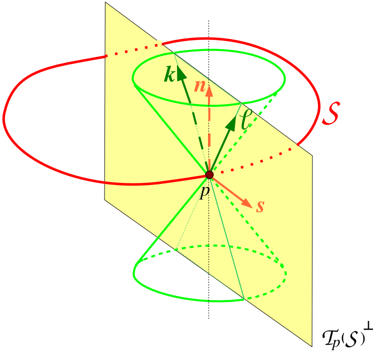


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*trapped surface* = **local** concept characterizing very strong gravitational fields

# Jean-Pierre's definition

$\mathcal{S}$ spherically symmetric (spacetime with spherical symmetry)
Define $R$ = areal radius of $\mathcal{S}$ : $A = 4\pi R^2$
*Definition:* "local event horizon" $\iff$ $\boldsymbol{\mathcal{L}_k} R < 0$ and $\boldsymbol{\mathcal{L}_\ell} R < 0$

In spherical symmetry, this is equivalent to $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} < 0$, i.e. the trapped surface condition

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# Link with apparent horizons

A closed spacelike 2-surface $\mathcal{S}$ is said to be outer trapped (resp. marginally outer trapped (MOTS)) iff [Hawking & Ellis 1973]

- the notions of *interior* and *exterior* of $\mathcal{S}$ can be defined (for instance spacetime asymptotically flat) $\Rightarrow$ $\boldsymbol{\ell}$ is chosen to be the *outgoing* null normal and $\boldsymbol{k}$ to be the *ingoing* one
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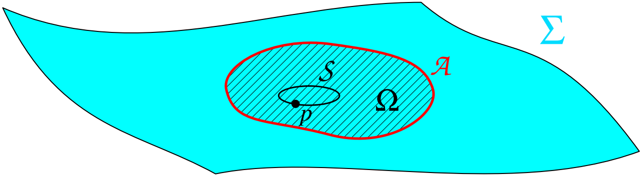



$\Sigma$: spacelike hypersurface extending to spatial infinity (Cauchy surface)

outer trapped region of $\Sigma$ : $\Omega$ = set of points $p \in \Sigma$ through which there is a outer trapped surface $\mathcal{S}$ lying in $\Sigma$

apparent horizon in $\Sigma$: $\mathcal{A}$ = connected component of the boundary of $\Omega$

*Proposition* [Hawking & Ellis 1973]: $\mathcal{A}$ smooth $\Longrightarrow$ $\mathcal{A}$ is a MOTS

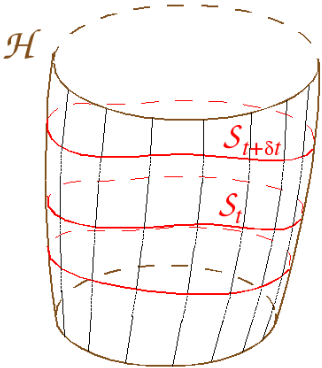
# Connection with singularities and black holes

*Proposition* [Penrose (1965)]:
provided that the weak energy condition holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a singularity in $(\mathscr{M}, \boldsymbol{g})$ (in the form of a future inextendible null geodesic)

*Proposition* [Hawking & Ellis (1973)]:
provided that the cosmic censorship conjecture holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a black hole $\mathcal{B}$ and $\mathcal{S} \subset \mathcal{B}$

# Local definitions of "black holes"

A hypersurface $\mathcal{H}$ of $(\mathscr{M}, \boldsymbol{g})$ is said to be




- a **future outer trapping horizon (FOTH)** iff
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces $(\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0)$
  (ii) $\mathcal{L}_{\boldsymbol{k}}\, \theta^{(\boldsymbol{\ell})} < 0$ (locally outermost trapped surf.)
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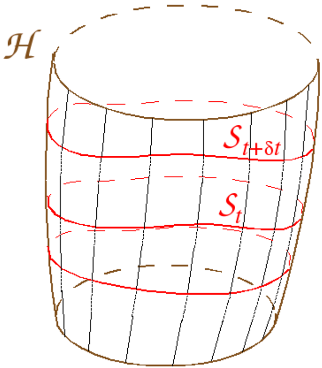


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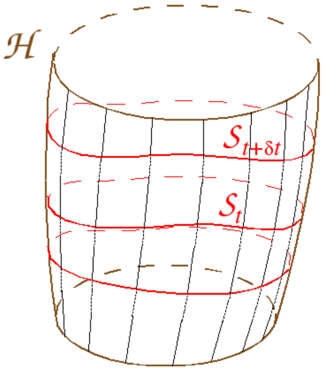


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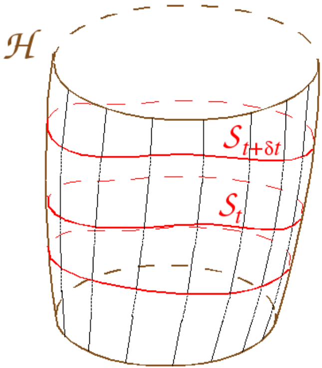


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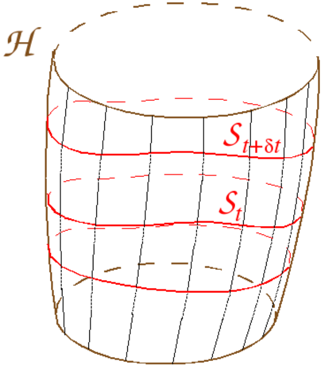


BH in equilibrium (e.g. Kerr) = IH
BH out of equilibrium = DH
generic BH = FOTH

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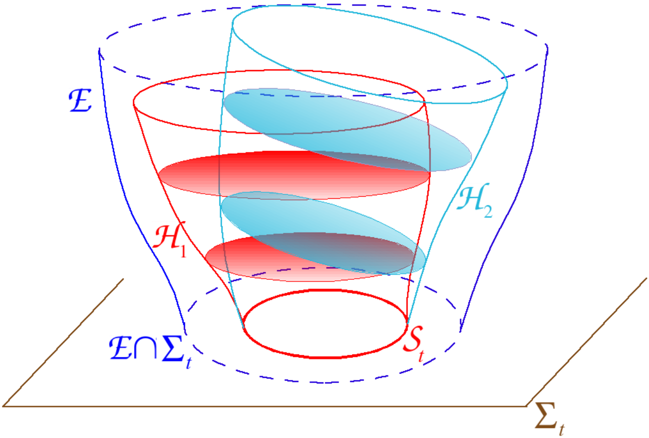
# Dynamics of these new horizons

The *trapping horizons* and *dynamical horizons* have their **own dynamics**, ruled by Einstein equations.
In particular, one can establish for them

- existence and (partial) uniqueness theorems
  [Andersson, Mars & Simon, PRL **95**, 111102 (2005)],
  [Ashtekar & Galloway, Adv. Theor. Math. Phys. **9**, 1 (2005)]
- first and second laws of black hole mechanics
  [Ashtekar & Krishnan, PRD **68**, 104030 (2003)], [Hayward, PRD **70**, 104027 (2004)]
- a viscous fluid bubble analogy ("membrane paradigm" as for the event horizon), leading to a Navier-Stokes-like equation and a **positive** bulk viscosity *(event horizon = negative bulk viscosity)*
  [Gourgoulhon, PRD **72**, 104007 (2005)], [Gourgoulhon & Jaramillo, PRD **74**, 087502 (2006)]

*Reviews:* [Ashtekar & Krishnan, Liv. Rev. Relat. **7**, 10 (2004)], [Booth, Can. J. Phys. **83**, 1073 (2005)], [Gourgoulhon & Jaramillo, Phys. Rep. **423**, 159 (2006)]

# Non-uniqueness of trapping horizons




*NB:* uniqueness in spherical symmetry

# Outline

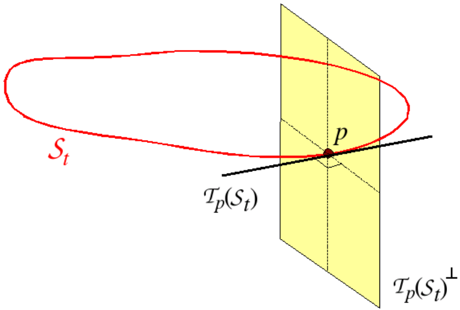
# Closed spacelike surfaces

$\mathcal{S}$ : closed (i.e. compact without boundary) spacelike 2-dimensional surface embedded in spacetime $(\mathscr{M}, \boldsymbol{g})$

$\mathcal{S}$ spacelike $\iff$ metric $\boldsymbol{q}$ induced by $\boldsymbol{g}$ is positive definite

$\boldsymbol{q}$ not degenerate $\Longrightarrow$ orthogonal decomposition of the tangent space at any $p \in \mathscr{M}$:

$$\mathcal{T}_p(\mathscr{M}) = \mathcal{T}_p(\mathcal{S}) \oplus \mathcal{T}_p(\mathcal{S})^{\perp}$$




$\boldsymbol{q}$: induced metric on $\mathcal{S}$, components: $q_{\alpha\beta}$

$\vec{\boldsymbol{q}}$: orthogonal projector onto $\mathcal{S}$, components: $q^{\alpha}{}_{\beta}$

# Projection operator $\vec{q}^*$

$\boldsymbol{A}$ : tensor of covariance type $(m, n)$
$\vec{\boldsymbol{q}}^* \boldsymbol{A}$ : tensor of same covariance type, defined by

$$(\vec{q}^* A)^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} := q^{\alpha_1}{}_{\mu_1} \cdots q^{\alpha_m}{}_{\mu_m} q^{\nu_1}{}_{\beta_1} \cdots q^{\nu_n}{}_{\beta_n} A^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n}$$

Remark: for a vector: $\vec{\boldsymbol{q}}^* \boldsymbol{v} = \vec{\boldsymbol{q}}(\boldsymbol{v})$
for a 1-form, $\vec{\boldsymbol{q}}^* \boldsymbol{\omega} = \boldsymbol{\omega} \circ \vec{\boldsymbol{q}}$

Definition: a tensor $\boldsymbol{A}$ is *tangent to* $\mathcal{S}$ iff $\vec{\boldsymbol{q}}^* \boldsymbol{A} = \boldsymbol{A}$.

# Expansion and shear along normal vectors

Let $\boldsymbol{v}$ be a vector field on $\mathscr{M}$, defined at least at $\mathcal{S}$ and everywhere normal to $\mathcal{S}$.
*NB: $\boldsymbol{v}$ is not assumed to be null*

**Deformation tensor of $\mathcal{S}$ along $\boldsymbol{v}$:** $\boxed{\boldsymbol{\Theta}^{(\boldsymbol{v})} := \vec{\boldsymbol{q}}^{*} \boldsymbol{\nabla} \underline{\boldsymbol{v}}}$ or $\Theta^{(\boldsymbol{v})}_{\alpha\beta} := \nabla_\nu v_\mu \, q^\mu{}_\alpha q^\nu{}_\beta$

$\boldsymbol{v}$ normal to a 2-surface $(\mathcal{S}) \Longrightarrow \boldsymbol{\Theta}^{(\boldsymbol{v})}$ is a symmetric bilinear form

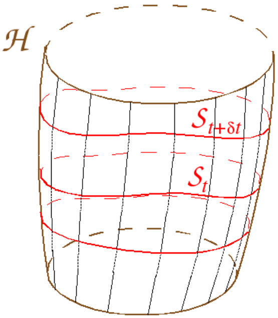
*Prop:* $\boldsymbol{\Theta}^{(\boldsymbol{v})} = \frac{1}{2} \vec{\boldsymbol{q}}^{*} \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \, \boldsymbol{q}$

Decomposition into traceless part (**shear** $\boldsymbol{\sigma}^{(\boldsymbol{v})}$) and trace part (**expansion** $\theta^{(\boldsymbol{v})}$ ):

$\boxed{\boldsymbol{\Theta}^{(\boldsymbol{v})} = \boldsymbol{\sigma}^{(\boldsymbol{v})} + \frac{1}{2} \theta^{(\boldsymbol{v})} \, \boldsymbol{q}}$ with $\theta^{(\boldsymbol{v})} := q^{\mu\nu} \Theta^{(\boldsymbol{v})}_{\mu\nu} = \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \ln \sqrt{q}$, $q := \det q_{ab}$

*Prop:* $\boxed{\boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \, {}^{\mathcal{S}}\!\boldsymbol{\epsilon} = \theta^{(\boldsymbol{v})} \, {}^{\mathcal{S}}\!\boldsymbol{\epsilon}}$ with ${}^{\mathcal{S}}\!\boldsymbol{\epsilon}$ surface element of $(\mathcal{S}, \boldsymbol{q})$ : ${}^{\mathcal{S}}\!\boldsymbol{\epsilon} = \sqrt{q} \, \mathbf{d}x^2 \wedge \mathbf{d}x^3$
$\Longrightarrow$ hence the name *expansion*

# Foliation of a hypersurface by spacelike 2-surfaces



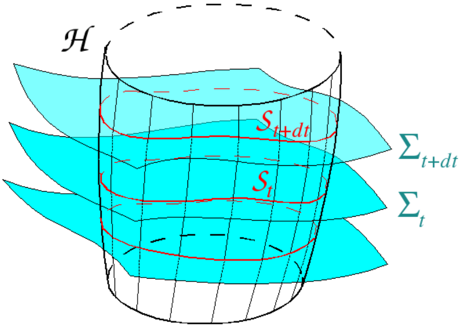

hypersurface $\mathcal{H}$ = submanifold of spacetime $(\mathscr{M}, \boldsymbol{g})$ of codimension 1

$\mathcal{H}$ can be $\begin{cases} \text{spacelike} \\ \text{null} \\ \text{timelike} \end{cases}$

$$\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$$

$\mathcal{S}_t$ = spacelike 2-surface

# Foliation of a hypersurface by spacelike 2-surfaces

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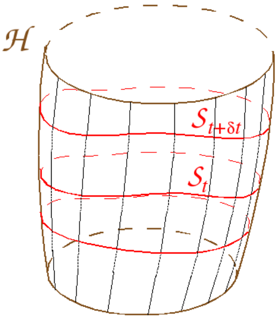
$\Longleftarrow$ 3+1 perspective

# Foliation of a hypersurface by spacelike 2-surfaces

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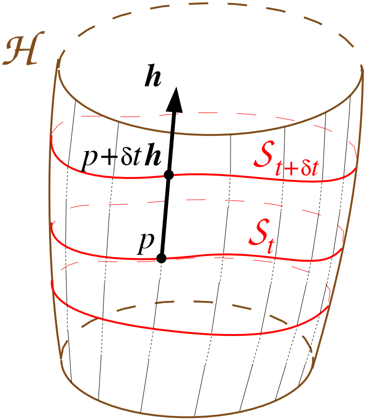
$$\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$$

$\mathcal{S}_t$ = spacelike 2-surface

intrinsic viewpoint adopted here (i.e. not relying on extra-structure such as a 3+1 foliation)

# Evolution vector on the horizon




Vector field $\boldsymbol{h}$ on $\mathcal{H}$ defined by

- (i) $\boldsymbol{h}$ is tangent to $\mathcal{H}$
- (ii) $\boldsymbol{h}$ is orthogonal to $\mathcal{S}_t$
- (iii) $\boldsymbol{\mathcal{L}_h}\, t = h^\mu \partial_\mu t = \langle \mathbf{d}t, \boldsymbol{h} \rangle = 1$

NB: (iii) $\Longrightarrow$ the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$

# Lie derivatives along $h$

Since the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$, so are their tangent vectors:

$$\forall \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t), \ \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t)$$

i.e. $\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}$ = internal operator on $\mathcal{T}(\mathcal{S}_t)$

Extension to 1-forms in $\mathcal{T}^*(\mathcal{S}_t)$:

$$\forall \boldsymbol{v} \in \mathcal{T}(\mathcal{S}_t), \quad \langle \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{\omega}, \boldsymbol{v} \rangle := \boldsymbol{\mathcal{L}}_{\boldsymbol{h}} \langle \boldsymbol{\omega}, \boldsymbol{v} \rangle - \langle \boldsymbol{\omega}, \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{v} \rangle.$$

Extension to any tensor $\boldsymbol{A}$ tangent to $\mathcal{S}_t$ by tensor products

Definition:

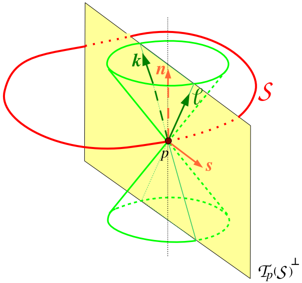
$$^{\mathcal{S}}\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{A} := \vec{\boldsymbol{q}}^{*} \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{A} = \vec{\boldsymbol{q}}^{*} \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \vec{\boldsymbol{q}}^{*} \boldsymbol{A}$$

# Norm of $\boldsymbol{h}$ and type of $\mathcal{H}$

Definition: $C := \frac{1}{2}\,\boldsymbol{h}\cdot\boldsymbol{h}$

$$\begin{array}{lcccl} \mathcal{H} \text{ is spacelike} & \iff & C > 0 & \iff & \boldsymbol{h} \text{ is spacelike} \\ \mathcal{H} \text{ is null} & \iff & C = 0 & \iff & \boldsymbol{h} \text{ is null} \\ \mathcal{H} \text{ is timelike} & \iff & C < 0 & \iff & \boldsymbol{h} \text{ is timelike.} \end{array}$$

# Frames normal to $\mathcal{S}_t$




Two natural types of choice for a vector basis of $\mathcal{T}_p(\mathcal{S}_t)^\perp$ :

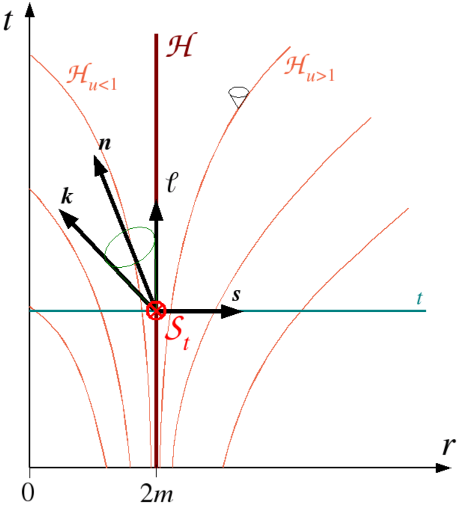
1. an orthonormal basis $(\boldsymbol{n}, \boldsymbol{s})$ ($\boldsymbol{n}$ = timelike, $\boldsymbol{s}$ = spacelike):
   $\boldsymbol{n}\cdot\boldsymbol{n} = -1, \quad \boldsymbol{s}\cdot\boldsymbol{s} = 1, \quad \boldsymbol{n}\cdot\boldsymbol{s} = 0$
2. a pair linearly independent future-directed null vectors $(\boldsymbol{\ell}, \boldsymbol{k})$:
   $\boldsymbol{\ell}\cdot\boldsymbol{\ell} = 0, \quad \boldsymbol{k}\cdot\boldsymbol{k} = 0, \quad \boldsymbol{\ell}\cdot\boldsymbol{k} =: -e^{\sigma}$

Degrees of freedom:

1. boost : $\begin{cases} \boldsymbol{n}' = \cosh\eta\,\boldsymbol{n} + \sinh\eta\,\boldsymbol{s} \\ \boldsymbol{s}' = \sinh\eta\,\boldsymbol{n} + \cosh\eta\,\boldsymbol{s} \end{cases}, \quad \eta \in \mathbb{R}$
2. rescaling : $\begin{cases} \boldsymbol{\ell}' = \lambda\,\boldsymbol{\ell}, & \lambda > 0 \\ \boldsymbol{k}' = \mu\,\boldsymbol{k}, & \mu > 0 \end{cases}$

Orthogonal projector: $\vec{\boldsymbol{q}} = \mathbf{1} + \langle \underline{\boldsymbol{n}}, .\rangle\,\boldsymbol{n} - \langle \underline{\boldsymbol{s}}, .\rangle\,\boldsymbol{s} = \mathbf{1} + e^{-\sigma}\langle \underline{\boldsymbol{k}}, .\rangle\,\boldsymbol{\ell} + e^{-\sigma}\langle \underline{\boldsymbol{\ell}}, .\rangle\,\boldsymbol{k}$

# Example of normal frames




$\mathcal{H}$ = event horizon of Schwarzschild black hole
$\mathcal{S}_t$ = slice of constant Eddington-Finkelstein time

# Second fundamental tensor of $\mathcal{S}_t$

Tensor $\mathcal{K}$ of type $(1,2)$ relating the covariant derivative of a vector tangent to $\mathcal{S}_t$ taken by the spacetime connection $\boldsymbol{\nabla}$ to that taken by the connection $\boldsymbol{\mathcal{D}}$ in $\mathcal{S}_t$ compatible with the induced metric $\boldsymbol{q}$:

$$\forall(\boldsymbol{u},\boldsymbol{v}) \in \mathcal{T}(\mathcal{S}_t)^2, \quad \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = \boldsymbol{\mathcal{D}}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\mathcal{K}}(\boldsymbol{u},\boldsymbol{v})$$

*Prop:*

$\mathcal{K}^{\alpha}{}_{\beta\gamma} = \nabla_{\mu} q^{\alpha}{}_{\nu}\, q^{\mu}{}_{\beta} q^{\nu}{}_{\gamma}$

$\mathcal{K}^{\alpha}{}_{\beta\gamma} = n^{\alpha}\, \Theta^{(\boldsymbol{n})}_{\beta\gamma} - s^{\alpha}\, \Theta^{(\boldsymbol{s})}_{\beta\gamma} = e^{-\sigma}\left(k^{\alpha}\, \Theta^{(\boldsymbol{\ell})}_{\beta\gamma} + \ell^{\alpha}\, \Theta^{(\boldsymbol{k})}_{\beta\gamma}\right)$

*Remark:* for a hypersurface of normal $\boldsymbol{n}$ and extrinsic curvature $\boldsymbol{K}$,

$\mathcal{K}^{\alpha}{}_{\beta\gamma} = -n^{\alpha} K_{\beta\gamma}$

# Normal fundamental forms

**Extrinsic geometry of $\mathcal{S}_t$ not entirely specified by $\mathcal{K}$ (contrary to the hypersurface case)**

$\mathcal{K}$ involves only the deformation tensors $\mathbf{\Theta}^{(\cdot)}$ of the normals to $\mathcal{S}_t \Longrightarrow \mathcal{K}$ encodes only the part of the variation of $\mathcal{S}_t$'s normals which is parallel to $\mathcal{S}_t$

Variation of the two normals with respect to each other: encoded by the **normal fundamental forms** (also called *external rotation coefficients* or *connection on the normal bundle*, or if $\mathcal{H}$ is null, *Hájíček 1-form*):

1. $\boxed{\mathbf{\Omega}^{(\boldsymbol{n})} := \boldsymbol{s} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{n}}$ or $\Omega^{(n)}_{\alpha} := s_\mu \nabla_\nu n^\mu \, q^\nu{}_\alpha$

   $\mathbf{\Omega}^{(\boldsymbol{s})} := \boldsymbol{n} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{s}$

2. $\boxed{\mathbf{\Omega}^{(\boldsymbol{\ell})} := \dfrac{1}{\boldsymbol{k} \cdot \boldsymbol{\ell}}\, \boldsymbol{k} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{\ell}}$ or $\Omega^{(\ell)}_{\alpha} := \dfrac{1}{k_\rho \ell^\rho} k_\mu \nabla_\nu \ell^\mu \, q^\nu{}_\alpha$

   $\mathbf{\Omega}^{(\boldsymbol{k})} := \dfrac{1}{\boldsymbol{k} \cdot \boldsymbol{\ell}}\, \boldsymbol{\ell} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{k}$

# Basic properties of the normal fundamental forms

From the definition: $\boldsymbol{\Omega}^{(s)} = -\boldsymbol{\Omega}^{(n)}$ and $\boldsymbol{\Omega}^{(k)} = -\boldsymbol{\Omega}^{(\ell)} + \boldsymbol{\mathcal{D}}\sigma$

Relation between the $(\boldsymbol{n}, \boldsymbol{s})$-type and the $(\boldsymbol{\ell}, \boldsymbol{k})$-type:
$\boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\Omega}^{(n)}$ $[\boldsymbol{\ell} = \boldsymbol{n} + \boldsymbol{s}]$ and $\boldsymbol{\Omega}^{(k)} = -\boldsymbol{\Omega}^{(n)}$ $[\boldsymbol{k} = \boldsymbol{n} - \boldsymbol{s}]$

## The normal fundamental forms are not unique

(contrary to the second fundamental tensor $\mathcal{K}$)
Dependence of the normal frame

1. $(\boldsymbol{n}, \boldsymbol{s}) \mapsto (\boldsymbol{n}', \boldsymbol{s}') \Longrightarrow$ $\boxed{\boldsymbol{\Omega}^{(n')} = \boldsymbol{\Omega}^{(n)} + \boldsymbol{\mathcal{D}}\eta}$
2. $(\boldsymbol{\ell}, \boldsymbol{k}) \mapsto (\boldsymbol{\ell}', \boldsymbol{k}') \Longrightarrow$ $\boxed{\boldsymbol{\Omega}^{(\ell')} = \boldsymbol{\Omega}^{(\ell)} + \boldsymbol{\mathcal{D}} \ln \lambda}$

# "Surface-gravity" 1-forms

If the vector fields $(\boldsymbol{\ell}, \boldsymbol{k})$ are extended away from $\mathcal{S}_t$, define the 1-form

$$\boxed{\boldsymbol{\kappa}^{(\boldsymbol{\ell})} := \frac{1}{\boldsymbol{k}\cdot\boldsymbol{\ell}}\,\boldsymbol{k}\cdot\boldsymbol{\nabla}_{\boldsymbol{p}}\,\boldsymbol{\ell}}$$

or $\kappa^{(\boldsymbol{\ell})}_{\alpha} := \frac{1}{k_\rho \ell^\rho} k_\mu \nabla_\nu \ell^\mu \, p^\nu{}_\alpha$

where $\boldsymbol{p}$ is the orthogonal projector complementary to $\vec{\boldsymbol{q}}$: $\mathbf{1} = \vec{\boldsymbol{q}} + \boldsymbol{p}$.

*NB:* Since $\boldsymbol{p}$ is a projector in a direction transverse to $\mathcal{S}_t$, the 1-form $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$ is not intrinsic to the 2-surface $\mathcal{S}_t$: it depends on the choice of $\boldsymbol{\ell}$ away from $\mathcal{S}_t$

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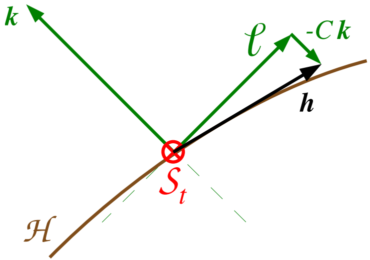
If $\boldsymbol{\ell}$ is extended along one of the two families of light rays emanating radially from $\mathcal{S}_t$, then $\boldsymbol{\ell}$ is pre-geodesic: $\boldsymbol{\nabla}_{\boldsymbol{\ell}}\,\boldsymbol{\ell} = \nu_{(\boldsymbol{\ell})}\,\boldsymbol{\ell}$, with the *inaffinity parameter* (*surface gravity* if $\boldsymbol{\ell}$ = null Killing vector of Kerr spacetime) given by the 1-form $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$ applied to $\boldsymbol{\ell}$:

$$\nu_{(\boldsymbol{\ell})} = \langle \boldsymbol{\kappa}^{(\boldsymbol{\ell})}, \boldsymbol{\ell} \rangle$$

# Normal null frame associated with the evolution vector




The foliation $(\mathcal{S}_t)_{t\in\mathbb{R}}$ entirely fixes the ambiguities in the choice of the null normal frame $(\boldsymbol{\ell}, \boldsymbol{k})$, via the evolution vector $\boldsymbol{h}$: there exists a unique normal null frame $(\boldsymbol{\ell}, \boldsymbol{k})$ such that

$$\boldsymbol{h} = \boldsymbol{\ell} - C\boldsymbol{k} \qquad \text{and} \qquad \boldsymbol{\ell}\cdot\boldsymbol{k} = -1$$

# Outline

# Concept of black hole viscosity

- Hartle and Hawking (1972, 1973): introduced the concept of **black hole viscosity** when studying the response of the *event horizon* to external perturbations
- Damour (1979): 2-dimensional **Navier-Stokes** like equation for the event horizon $\Longrightarrow$ *shear viscosity* and *bulk viscosity*
- Thorne and Price (1986): **membrane paradigm** for black holes

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Shall we restrict the analysis to the event horizon ?

Can we extend the concept of viscosity to the local characterizations of black hole recently introduced, i.e. future outer trapping horizons and dynamical horizons ?

NB: *event horizon* = null hypersurface
*future outer trapping horizon* = null or spacelike hypersurface
*dynamical horizon* = spacelike hypersurface

# Navier-Stokes equation in Newtonian fluid dynamics

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j \nabla_j v^i\right) = -\nabla^i P + \mu \Delta v^i + \left(\zeta + \frac{\mu}{3}\right) \nabla^i(\nabla_j v^j) + f^i$$

or, in terms of fluid momentum density $\pi_i := \rho v_i$,

$$\boxed{\frac{\partial \pi_i}{\partial t} + v^j \nabla_j \pi_i + \theta \pi_i = -\nabla_i P + 2\mu \nabla^j \sigma_{ij} + \zeta \nabla_i \theta + f_i}$$

where $\theta$ is the fluid expansion:

$$\theta := \nabla_j v^j$$

and $\sigma_{ij}$ the velocity shear tensor:

$$\sigma_{ij} := \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right) - \frac{1}{3}\theta\, \delta_{ij}$$

$P$ is the pressure, $\mu$ the shear viscosity, $\zeta$ the bulk viscosity and $f_i$ the density of external forces

# Original Damour-Navier-Stokes equation

*Hyp:* $\mathcal{H}$ = null hypersurface (particular case: black hole **event horizon**)
Then $\boldsymbol{h} = \boldsymbol{\ell}$ ($C = 0$) reminder

Damour (1979) has derived from Einstein equation the relation

$$\boxed{{}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{\ell})} \boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}} \nu_{(\boldsymbol{\ell})} - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\boldsymbol{\ell})} + \frac{1}{2} \boldsymbol{\mathcal{D}} \theta^{(\boldsymbol{\ell})} + 8\pi \vec{\boldsymbol{q}}^{*} \boldsymbol{T} \cdot \boldsymbol{\ell}}$$

or equivalently

$$\boxed{{}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{\pi} + \theta^{(\boldsymbol{\ell})} \boldsymbol{\pi} = -\boldsymbol{\mathcal{D}} P + 2\mu \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\boldsymbol{\ell})} + \zeta \boldsymbol{\mathcal{D}} \theta^{(\boldsymbol{\ell})} + \boldsymbol{f}}$$

with $\boldsymbol{\pi} := -\dfrac{1}{8\pi} \boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ momentum surface density

$P := \dfrac{\nu_{(\boldsymbol{\ell})}}{8\pi}$ pressure

$\mu := \dfrac{1}{16\pi}$ shear viscosity

$\zeta := -\dfrac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{\boldsymbol{q}}^{*} \boldsymbol{T} \cdot \boldsymbol{\ell}$ external force surface density ($\boldsymbol{T}$ = stress-energy tensor)

# Original Damour-Navier-Stokes equation (con't)

Introducing a coordinate system $(t, x^1, x^2, x^3)$ such that

- $t$ is compatible with $\boldsymbol{\ell}$: $\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, t = 1$
- $\mathcal{H}$ is defined by $x^1 = \text{const}$, so that $x^a = (x^2, x^3)$ are coordinates spanning $\mathcal{S}_t$

then

$$\boldsymbol{\ell} = \frac{\partial}{\partial t} + \boldsymbol{V}$$

with $\boldsymbol{V}$ tangent to $\mathcal{S}_t$: velocity of $\mathcal{H}$'s null generators with respect to the coordinates $x^a$ [Damour, PRD **18**, 3598 (1978)].

Then

$$\theta^{(\boldsymbol{\ell})} = \mathcal{D}_a V^a + \frac{\partial}{\partial t} \ln \sqrt{q} \qquad q := \det q_{ab}$$

$$\sigma^{(\boldsymbol{\ell})}_{ab} = \frac{1}{2}\left(\mathcal{D}_a V_b + \mathcal{D}_b V_a\right) - \frac{1}{2}\theta^{(\boldsymbol{\ell})}\, q_{ab} + \frac{1}{2}\frac{\partial q_{ab}}{\partial t}$$

compare

# Negative bulk viscosity of event horizons

From the Damour-Navier-Stokes equation, $\boxed{\zeta = -\frac{1}{16\pi} < 0}$

This negative value would yield to a *dilation or contraction instability* in an ordinary fluid

It is in agreement with the tendency of a null hypersurface to continually contract or expand

The event horizon is stabilized by the teleological condition imposing its expansion to vanish in the far future (equilibrium state reached)

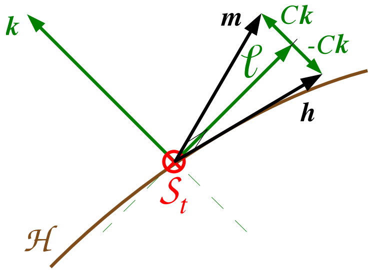
# Generalization to the non-null case

**Starting remark: in the null case, $\ell$ plays two different roles:**

- evolution vector along $\mathcal{H}$ (e.g. term ${}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}$ )
- normal to $\mathcal{H}$ (e.g. term $\vec{\boldsymbol{q}}^* \boldsymbol{T} \cdot \boldsymbol{\ell}$)

When $\mathcal{H}$ is no longer null, these two roles have to be taken by two different vectors:

- evolution vector: obviously $\boldsymbol{h}$ reminder
- vector normal to $\mathcal{H}$: a natural choice is $\boxed{\boldsymbol{m} := \boldsymbol{\ell} + C\boldsymbol{k}}$



# Generalized Damour-Navier-Stokes equation

Starting point of the calculation: contracted Ricci identity applied to the vector $\boldsymbol{m}$ and projected onto $\mathcal{S}_t$:

$$(\nabla_\mu \nabla_\nu m^\mu - \nabla_\nu \nabla_\mu m^\mu)\, q^\nu{}_\alpha = R_{\mu\nu} m^\mu q^\nu{}_\alpha$$

Final result:

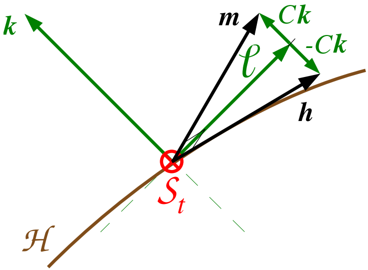
$$\boxed{{}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}}\langle \boldsymbol{\kappa}^{(\boldsymbol{\ell})}, \boldsymbol{h}\rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{m})} - \theta^{(\boldsymbol{k})}\boldsymbol{\mathcal{D}}C + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{m}}$$

- $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ : normal fundamental form of $\mathcal{S}_t$ associated with null normal $\boldsymbol{\ell}$ reminder
- $\theta^{(\boldsymbol{h})}$, $\theta^{(\boldsymbol{m})}$ and $\theta^{(\boldsymbol{k})}$: expansion scalars of $\mathcal{S}_t$ along the vectors $\boldsymbol{h}$, $\boldsymbol{m}$ and $\boldsymbol{k}$ respectively reminder
- $\boldsymbol{\mathcal{D}}$ : covariant derivative within $(\mathcal{S}_t, \boldsymbol{q})$
- $\boldsymbol{\kappa}^{(\boldsymbol{\ell})}$ : "surface-gravity" 1-form associated with the null vector $\boldsymbol{\ell}$ reminder
- $\boldsymbol{\sigma}^{(\boldsymbol{m})}$ : shear tensor of $\mathcal{S}_t$ along the vector $\boldsymbol{m}$ reminder
- $C$ : half the scalar square of $\boldsymbol{h}$ reminder

# Null limit

In the null limit,

$$\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell} \qquad \text{and} \qquad C = 0$$




and we recover the original Damour-Navier-Stokes equation:

$$\boxed{{}^{\mathcal{S}}\mathcal{L}_{\boldsymbol{\ell}}\, \boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)}\boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\nu_{(\ell)} - \boldsymbol{\mathcal{D}}\cdot\vec{\boldsymbol{\sigma}}^{(\ell)} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\ell)} + 8\pi\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}}$$

# Case of future trapping horizons

Definition [Hayward, PRD **49**, 6467 (1994)] : $\mathcal{H}$ is a **future trapping horizon** iff $\theta^{(\ell)} = 0$ and $\theta^{(k)} < 0$.

The generalized Damour-Navier-Stokes equation reduces then to

$$\boxed{{}^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\ell)} + \theta^{(h)}\, \boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\langle \boldsymbol{\kappa}^{(\ell)}, \boldsymbol{h}\rangle - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(m)} - \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(h)} - \theta^{(k)}\boldsymbol{\mathcal{D}}C + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{m}}$$

*NB:* Notice the change of sign in the $-\dfrac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(h)}$ term with respect to the original Damour-Navier-Stokes equation compare

# Viscous fluid form

$$\boxed{{}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\,\boldsymbol{\pi} + \theta^{(\boldsymbol{h})}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + \frac{1}{8\pi}\boldsymbol{\mathcal{D}}\cdot\vec{\boldsymbol{\sigma}}^{(\boldsymbol{m})} + \zeta\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{h})} + \boldsymbol{f}}$$

with $\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ momentum surface density

$P := \frac{\kappa}{8\pi}$ pressure

$\frac{1}{8\pi}\boldsymbol{\sigma}^{(\boldsymbol{m})}$ shear stress tensor

$\zeta := \frac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{q}^{*}\boldsymbol{T}\cdot\boldsymbol{m} + \frac{\theta^{(\boldsymbol{k})}}{8\pi}\boldsymbol{\mathcal{D}}C$ external force surface density

Similar to the Damour-Navier-Stokes equation for an event horizon hyperlink, except for

- no Newtonian-fluid relation between *stress* and *strain*: $\boldsymbol{\sigma}^{(\boldsymbol{m})} \neq 2\mu\boldsymbol{\sigma}^{(\boldsymbol{h})}$
- positive bulk viscosity

This positive value of bulk viscosity shows that FOTHs and DHs behave as "ordinary" physical objects, in perfect agreement with their local nature

# Generalized angular momentum

Definition [Booth & Fairhurst, CQG **22**, 4545 (2005)]: Let $\varphi$ be a vector field on $\mathcal{H}$ which

- is tangent to $\mathcal{S}_t$
- has closed orbits
- has vanishing divergence with respect to the induced metric: $\mathcal{D} \cdot \varphi = 0$

For dynamical horizons, $\theta^{(h)} \neq 0$ and there is a unique choice of $\varphi$ as the generator (conveniently normalized) of the curves of constant $\theta^{(h)}$ [Hayward, PRD **74**, 104013 (2006)]

The *generalized angular momentum associated with* $\varphi$ is then defined by

$$\boxed{J(\varphi) := -\frac{1}{8\pi} \oint_{\mathcal{S}_t} \langle \mathbf{\Omega}^{(\ell)}, \varphi \rangle \, {}^{\mathcal{S}}\!\epsilon,}$$

*Remark 1:* does not depend upon the choice of null vector $\ell$, thanks to the divergence-free property of $\varphi$

*Remark 2:*

- coincides with Ashtekar & Krishnan's definition for a dynamical horizon
- coincides with Brown-York angular momentum if $\mathcal{H}$ is timelike and $\varphi$ a Killing vector

# Angular momentum flux law

Under the supplementary hypothesis that $\varphi$ is transported along the evolution vector $\boldsymbol{h}$ : $\mathcal{L}_{\boldsymbol{h}}\,\boldsymbol{\varphi} = 0$, the generalized Damour-Navier-Stokes equation leads to

$$\boxed{\frac{d}{dt}J(\boldsymbol{\varphi}) = -\oint_{\mathcal{S}_t} \boldsymbol{T}(\boldsymbol{m},\boldsymbol{\varphi})\,{}^{\mathcal{S}}\boldsymbol{\epsilon} - \frac{1}{16\pi}\oint_{\mathcal{S}_t}\left[\vec{\vec{\boldsymbol{\sigma}}}^{(\boldsymbol{m})} : \mathcal{L}_{\boldsymbol{\varphi}}\,\boldsymbol{q} - 2\theta^{(\boldsymbol{k})}\boldsymbol{\varphi}\cdot\boldsymbol{\mathcal{D}}C\right]{}^{\mathcal{S}}\boldsymbol{\epsilon}}$$

[Gourgoulhon, PRD **72**, 104007 (2005)]

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Two interesting limiting cases:

- $\mathcal{H}$ = null hypersurface : $C = 0$ and $\boldsymbol{m} = \boldsymbol{\ell}$ :

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  i.e. Eq. (6.134) of the *Membrane Paradigm* book (Thorne, Price & MacDonald 1986)

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- $\mathcal{H}$ = future trapping horizon :

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# Outline

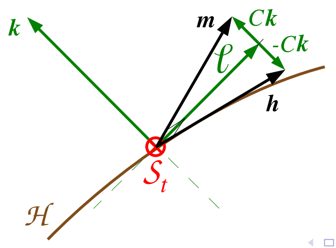
# Starting point

From the Einstein equation, one can derive the following evolution law for any foliated hypersurface $\mathcal{H}$ [Gourgoulhon & Jaramillo, PRD **74**, 087502 (2006)] :

$$\begin{aligned}\mathcal{L}_{\boldsymbol{h}}\,\theta^{(\boldsymbol{m})} &= \kappa\,\theta^{(\boldsymbol{h})} - \frac{1}{2}\theta^{(\boldsymbol{h})}\theta^{(\boldsymbol{m})} - \boldsymbol{\sigma}^{(\boldsymbol{h})} : \boldsymbol{\sigma}^{(\boldsymbol{m})} - 8\pi \boldsymbol{T}(\boldsymbol{m},\boldsymbol{h}) \\ &\quad + \theta^{(\boldsymbol{k})}\mathcal{L}_{\boldsymbol{h}}\,C + \boldsymbol{\mathcal{D}}\cdot\left(2C\vec{\boldsymbol{\Omega}}^{(\boldsymbol{\ell})} - \vec{\boldsymbol{\mathcal{D}}}C\right)\end{aligned}$$

where $\kappa$ is the component along $\boldsymbol{\ell}$ of the "acceleration" of $\boldsymbol{h}$ in the decomposition

$$\boldsymbol{\nabla}_{\boldsymbol{h}}\boldsymbol{h} = \kappa\,\boldsymbol{\ell} + (C\kappa - \mathcal{L}_{\boldsymbol{h}}\,C)\boldsymbol{k} - \boldsymbol{\mathcal{D}}C$$



# Two special cases

- **null hypersurface (event horizon)** : $\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell}$ and $C = 0$:

$$\mathcal{L}_{\boldsymbol{\ell}}\, \theta^{(\boldsymbol{\ell})} + (\theta^{(\boldsymbol{\ell})})^2 - \kappa\, \theta^{(\boldsymbol{\ell})} = \frac{1}{2}(\theta^{(\boldsymbol{\ell})})^2 - \boldsymbol{\sigma}^{(\boldsymbol{\ell})} : \boldsymbol{\sigma}^{(\boldsymbol{\ell})} - 8\pi \boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell})$$

$\longrightarrow$ this is the *null Raychaudhuri equation*

- **FOTH** : $\theta^{(\boldsymbol{\ell})} = 0 \Rightarrow \theta^{(\boldsymbol{m})} = -\theta^{(\boldsymbol{h})}$ :

$$\begin{aligned} \mathcal{L}_{\boldsymbol{h}}\, \theta^{(\boldsymbol{h})} + (\theta^{(\boldsymbol{h})})^2 + \kappa\, \theta^{(\boldsymbol{h})} \quad &= \quad \frac{1}{2}(\theta^{(\boldsymbol{h})})^2 + \boldsymbol{\sigma}^{(\boldsymbol{h})} : \boldsymbol{\sigma}^{(\boldsymbol{m})} + 8\pi \boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h}) \\ & \quad -\theta^{(\boldsymbol{k})} \mathcal{L}_{\boldsymbol{h}}\, C + \boldsymbol{\mathcal{D}} \cdot \left( \vec{\boldsymbol{\mathcal{D}}} C - 2C \vec{\boldsymbol{\Omega}}^{(\boldsymbol{\ell})} \right) \end{aligned}$$

Notice the change of signs between the two cases

# Energy equation

For a event horizon, Price and Thorne (1986) have defined the surface energy density as $\varepsilon := -\frac{1}{8\pi}\theta^{(\ell)}$

By analogy, define the surface energy density of a FOTH as $\boxed{\varepsilon := -\frac{1}{8\pi}\theta^{(\boldsymbol{m})}}$

Then the above evolution equation becomes

$$\boxed{\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\,\varepsilon + (\varepsilon + P)\theta^{(\boldsymbol{h})} = \frac{1}{8\pi}\boldsymbol{\sigma}^{(\boldsymbol{h})}{:}\boldsymbol{\sigma}^{(\boldsymbol{m})} + \zeta(\theta^{(\boldsymbol{h})})^2 - \boldsymbol{\mathcal{D}}\cdot\boldsymbol{Q} + \mathcal{R}}$$

[Gourgoulhon & Jaramillo, PRD **74**, 087502 (2006)]

with $P := \frac{\kappa}{8\pi}$ pressure, $\frac{1}{8\pi}\boldsymbol{\sigma}^{(\boldsymbol{m})}$ shear stress tensor

$\boldsymbol{\sigma}^{(\boldsymbol{h})}$ shear strain tensor, $\boxed{\zeta := \frac{1}{16\pi} > 0}$ bulk viscosity

$\boldsymbol{Q} := \frac{1}{4\pi}\left[C\vec{\boldsymbol{\Omega}}^{(\ell)} - \frac{1}{2}\vec{\boldsymbol{\mathcal{D}}}C\right]$ heat flux

$\mathcal{R} = \boldsymbol{T}(\boldsymbol{m},\boldsymbol{h}) - \frac{\theta^{(\boldsymbol{k})}}{8\pi}\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\,C$ external energy production rate

We recover the positiveness of the bulk viscosity for a FOTH